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The FP_m-Conjecture for a Class of Metabelian Groups

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We prove that if $A \to G \to Q$ is a short exact sequence of groups where G is finitely generated, A and Q are abelian, A is a $\mathbb{Z}$-torsion Krull dimension one $\mathbb{Z}Q$-module via conjugation then the FP_m-Conjecture holds. In general if G is of type FP_m and either the extension is split or A is $\mathbb{Z}$-torsion we show that A is m-tame as $\mathbb{Z}Q$-module. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let G be a finitely generated metabelian group with an abelian normal subgroup A and an abelian quotient $Q \simeq G/A$. Then A is a finitely generated (left) $\mathbb{Z}Q$-module, where Q acts via conjugation. One geometrical invariant of the $\mathbb{Z}Q$-module A is

$$\Sigma_A(Q) = \left\{ \chi \colon Q \to \mathbb{R} \mid \chi \neq 0,\ A \text{ is finitely generated as } \mathbb{Z}Q_\chi\text{-module} \right\} / \sim ,$$

where $Q_\chi = \{q \in Q : \chi(q) \geq 0\}$ and $\sim$ is an equivalence relation in $\operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$ with equivalence classes $\mathbb{R}^+\chi$, where $\chi \in \operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$. If n is the rank of Q then $\Sigma_A(Q)$ can be viewed as a subset of the unit sphere S^{n-1} in $\mathbb{R}^n \simeq \operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$. This presentation of $\Sigma_A(Q)$ as a subset of S^{n-1} depends on the chosen isomorphism between $\mathbb{R}^n$ and $\operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$. We assume from now on that such an isomorphism is fixed and consider $\Sigma_A(Q)$ as a subset of S^{n-1}.

The $\mathbb{Z}Q$-module A is said to be m-tame if any characters $\chi_1, \dots, \chi_m$ lying in

$$\Delta_A(Q) = \left\{ \chi \colon Q \to \mathbb{R} \mid \chi \neq 0,\ A \text{ is not finitely generated as } \mathbb{Z}Q_\chi\text{-module} \right\} \subseteq \mathbb{R}^n$$

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are contained in an open half subspace of $\mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R}) \simeq \mathbb{R}^n$. The class of 2-tame $\mathbb{Z}Q$-modules was first considered by Bieri and Strebel in [5].

A group G is said to be of type FP_m if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to \mathbb{Z}$$

with P_i finitely generated as $\mathbb{Z}G$-module for $i \leq m$. In [5] Bieri and Strebel prove that A is 2-tame if and only if G is finitely presented and show that this happens precisely when G is of type FP_2. Bieri and Groves [3] show that if G is of type FP_m and A is of exponent p for some prime integer p then A is m-tame as $\mathbb{Z}Q$-module. In [2] Bieri suggests the following conjecture

FP_m-Conjecture. G is of type FP_m if and only if A is m-tame.

In the case where G is a finitely generated metabelian group of finite Prufer rank, i.e., there is an integer d such that every finitely generated subgroup may be generated by at most d-generators, the FP_m-Conjecture is proved by Åberg [1]. In [11] Noskov generalizes Åberg's proof and shows that if A is torsion-free, G is a split extension of A by Q, and G is of type FP_m then A is m-tame.

One of the main results obtained in the paper is the following theorem.

THEOREM A. *Let G be a finitely generated group with a normal subgroup A and an abelian quotient $Q \simeq G/A$ such that A is a torsion abelian group and has Krull dimension one as $\mathbb{Z}Q$-module. If A is m-tame as $\mathbb{Z}Q$-module then G is of type FP_m.*

In order to prove Theorem A we construct a CW-complex Y, acted on by G with the following properties:

(i) Y/G is a compact space;

(ii) Y is $(m-1)$-acyclic;

(iii) the stabilizers in G of cells in Y are polycyclic.

Then we can apply the following criterion due to K. Brown.

THEOREM [9]. *Suppose G is a group acting on $(m-1)$-acyclic CW-complex Y. Let D be a directed index set and let $\{Y_\alpha\}_{\alpha \in D}$ be a filtration of Y by G-subcomplexes Y_α such that Y_α has finite m-skeleton $\operatorname{mod} G$ for all $\alpha \in D$. Furthermore, assume that every stabilizer G_e, where e is an i-cell of Y, is of type FP_{m-i}, $i \leq m$. Then G is of type FP_m if and only if the filtration $\{Y_\alpha\}_{\alpha \in D}$ is essentially $(m-1)$-acyclic (i.e., for every $\alpha \in D$ there exists $\beta \in D$, $\alpha \leq \beta$ such that $H_i(Y_\alpha) \to H_i(Y_\beta)$ is the trivial homomorphism for each $i \leq m-1$).*

Remark. In the particular case when the set D contains only one element $Y = Y_\alpha$ and the stabilizers are polycyclic, this being exactly the situation we will meet later, Åberg notices in [1] that if Y is $(m-1)$-acyclic then G is of type FP_m.

The construction of the CW-complex Y resembles the construction given in [1] but we do not require that G be a split extension of A by Q. The complex Y is defined as a subspace of the product X of some G-trees X_v and the G-trees X_v are similar to the valuation trees considered in the split case in [1]. The tree X_v is the tree associated to a Cayley graph of G and a discrete character v of Q. Details about this construction can be found in [7]; in 2.1 we explain all the properties of X_v we will need later.

We generalize one of the main ideas of the proof of the FP_m-Conjecture for metabelian groups of finite Prufer rank given by Åberg [1] and prove the following theorem

THEOREM B. *Let $A \to G \to Q$ be a short exact sequence of groups with G finitely generated, A and Q abelian, and either the extension is split or A is a torsion group. If G is of type FP_m then A is m-tame as $\mathbb{Z}Q$-module.*

As a consequence of Theorems A and B we obtain

COROLLARY C. *If G is a finitely generated group which is an extension of a $\mathbb{Z}$-torsion Krull dimension one $\mathbb{Z}Q$-module A by an abelian group Q then the FP_m-Conjecture holds.*

Recently K. U. Bux [10] has proved Corollary C in the case where A is an S-arithmetic ring of prime characteristic and the extension is split. He uses methods from algebraic number theory to prove that the group G acts cocompactly on a space of Åberg's type.

The paper is structured as follows: in 2.1 we define the G-tree X_v for a discrete character v of Q, in 2.2 we consider a complex of Åberg's type Y, the set of characters needed in the definition of the space Y is described in 2.3, in 2.4 we show that Y/G is a compact space, and finally in 2.5 we prove that the stabilizers in G of cells in Y are polycyclic.

The proof of Theorem B is discussed in Section 3. In 3.2 and 3.3 we define an element of $H^0(Q', (\mathbb{Z}G^{\mathbb{N}})_A)$ and in the case where A is not m-tame we show in 3.4 that the image of this element in $H_0(Q'', (\mathbb{Z}G^{\mathbb{N}})_A)$ is non-trivial, where Q' and Q'' are subgroups of Q with $Q' \times Q'' \simeq Q$. The last will contradict the fact that G is of type FP_m.

Since the FP_m-Conjecture holds for a group G if and only if it holds for a subgroup of finite index in G, we assume from now on that $Q \simeq G/A$ is free abelian of rank n. Therefore $Q = \mathbb{Z}^n \subset \mathbb{R}^n$, and from now on we consider the vector space $\mathbb{R}^n$ equipped with the standard inner product.

2. PROOF OF THEOREM A

2.1. *Trees Deriving from Cayley Graphs*

Let G be a finitely generated metabelian group with an abelian normal subgroup A and an abelian quotient $Q \simeq G/A \simeq \mathbb{Z}^n$. Let v be a discrete character of Q with $v(Q) = \mathbb{Z}$. It is shown in [7] that to every Cayley graph Γ corresponding to a finite set of generators of G there can be assigned a tree Γ_v, which depends on the character v. Here we give a detailed construction of a tree X_v that is in fact Γ_v for a suitably chosen set of generators of G and the corresponding Cayley graph Γ.

Set $Q_v = \{q \in Q : v(q) \geq 0\}$ and let $q_{v,1}, \ldots, q_{v,n-1}$ be generators of $\operatorname{Ker} v$. Assume q_v is an element of Q such that $v(q_v) = 1$. Let A be generated as $\mathbb{Z}Q$-module by the elements $a_1, \ldots, a_s$ and let π be a lifting of the projection map $G \to G/A$. We define A_v to be the $\mathbb{Z}Q_v$-submodule of A generated by $a_1, \ldots, a_s$ and G_v to be the subgroup of G generated by $q_v^{\beta_v} A_v$ and $\pi(q_{v,i})$ for $1 \leq i \leq n-1$, where β_v is a negative integer such that all the commutators of $\pi(q_v)^{\pm 1}$, $\pi(q_{v,i})^{\pm 1}$ for $1 \leq i \leq n-1$ lie in $q_v^{\beta_v} A_v$. Note that if A is finitely generated as $\mathbb{Z}Q_v$-module then by [5, Proposition 2.1] $A_v = A$, and if A is not finitely generated as $\mathbb{Z}Q_v$-module then $A_v \neq A$ and $\bigcup_{z \in \mathbb{Z}} q_v^z A_v = A$.

Since $[\pi(q_{v,j})^{\pm 1}, \pi(q_{v,i})^{\pm 1}] \in q_v^{\beta_v} A_v$ we have $G_v \cap A = q_v^{\beta_v} A_v$. At the same time $\pi(q_v)\pi(q_{v,i})^{\pm 1}\pi(q_v)^{-1} \in q_v^{\beta_v} A_v \pi(q_{v,i})^{\pm 1} \subseteq G_v$ and for $a \in q_v^{\beta_v} A_v$ we have $\pi(q_v) a \pi(q_v)^{-1} = q_v a \in q_v^{\beta_v + 1} A_v \subseteq q_v^{\beta_v} A_v$. So we get

$$\pi(q_v) G_v \pi(q_v)^{-1} \subseteq G_v. \tag{1}$$

We define G_v^+ to be the submonoid of G generated by G_v and $\pi(q_v)$. By (1)

$$G_v^+ = \bigcup_{z \geq 0} G_v \pi(q_v)^z. \tag{2}$$

Let T_0 be the G-set G/G_v with partial order given by $gG_v \leq hG_v$ if and only if $g^{-1}h \in G_v^+$. The character $\alpha_v \colon G \to \mathbb{Z}$ which extends the character v factors through a map $\chi_v \colon T_0 \to \mathbb{Z}$.

We call two points α, β of T_0 neighbours if $\alpha \leq \beta$, $\chi_v(\beta) - \chi_v(\alpha) = 1$, or $\beta \leq \alpha$, $\chi_v(\alpha) - \chi_v(\beta) = 1$. We link every two neighbours $\alpha \leq \beta$ with a closed unit interval and write $(1-t)\alpha + t\beta$ for the point of this interval having distances t and $1-t$ to α and β, respectively. The constructed intervals intersect only at their ends or do not intersect. We get a graph T with vertices T_0 and edges the constructed unit intervals between neighbours of T_0. We extend $\chi_v \colon T_0 \to \mathbb{Z}$ to an $\mathbb{R}$-linear map $\chi_v \colon T \to \mathbb{R}$; in other words $\chi_v((1-t)\alpha + t\beta) = (1-t)\chi_v(\alpha) + t\chi_v(\beta) = \chi_v(\alpha) + t$ where $\alpha \leq \beta$ and $\chi_v(\beta) - \chi_v(\alpha) = 1$.

If $\alpha \leq \beta$ are neighbours then their images $g * \alpha$ and $g * \beta$ under the action of an element g of G are neighbours as well and $g * \alpha \leq g * \beta$. We make G act on T by $g * ((1-t)\alpha + t\beta) = (1-t)(g * \alpha) + t(g * \beta)$ using the G-action on T_0. We write X_v for the graph T.

For any $g \in G$ we define $L_{g,v}$ to be the line in X_v spanned by the vertices $\{g\pi(q_v)^z G_v\colon z \in \mathbb{Z}\}$. Note that for any integer z there is an edge in X_v from $g\pi(q_v)^z G_v$ to $g\pi(q_v)^{z+1}G_v$. The restriction $\chi_v\colon L_{g,v} \to \mathbb{R}$ is a bijection. Since $G = \bigcup_{z \in \mathbb{Z}} A\pi(q_v)^z G_v$ and the edge between $g\pi(q_v)^z G_v$ and $g\pi(q_v)^{z+1}G_v$ is the unique one going to $g\pi(q_v)^{z+1}G_v$ (i.e., if $h_1 G_v < h_2 G_v$ are neighbours then $h_1 G_v = h_2 \pi(q_v)^{-1} G_v$) we get

$$X_v = \bigcup_{a \in A} L_{a,v}.$$

We write $[(a, r)]$ where $a \in A$ and $r \in \mathbb{R}$ for the unique element of $L_{a,v}$ whose image under χ_v is r. By the definition of the G-action on X_v we have $b * [(a, r)] = [(b + a, r)]$ where b, a are any elements of A and r is any real number.

Remark. The above description of X_v resembles the valuation tree associated to G and a valuation of A, extending the character v, in the case where A is an integral domain and 1-generated $\mathbb{Z}Q$-module, and G is a split extension of A by Q [1, Chap. 3.2].

LEMMA 2.1. (i) *For every $a, b \in A$ we have $L_{a,v} \cap L_{b,v} = \{[(a, r)] : r \leq$ some integer depending on $a, b\}$ or $L_{a,v} = L_{b,v}$. In particular X_v is a tree.*

(ii) *The graph X_v can be represented as a union of ascending rays T_i, $i \geq 2$ (i.e., $\chi_v\colon T_i \to \mathbb{R}$ is injective and the image of this map is $[r_i, \infty)$ for some real number r_i) and a line T_1 (where $\chi_v\colon T_1 \to \mathbb{R}$ is a bijection) such that for every j we have $(\bigcup_{i \leq j} T_i) \cap T_{j+1} = \{w_j\}$ for a vertex w_j of X_v and $\chi_v(T_{j+1}) = [\chi_v(w_j), \infty)$.*

Proof. (i) If a, b are elements of A and z is an integer then $[(a, z)] = [(b, z)]$ if and only if $q_v^{-z}(a - b) \in G_v \cap A = q_v^{\beta_v} A_v$. Let $z_0 = \sup\{z \in \mathbb{Z}\colon a - b \in q_v^{z+\beta_v} A_v\}$. If $z_0 = \infty$ we have $L_{a,v} = L_{b,v}$ and if $z \leq z_0 < \infty$ then $a - b \in q_v^{z_0+\beta_v} A_v \subseteq q_v^{z+\beta_v} A_v$. Thus $L_{a,v} \cap L_{b,v} = \{[(a, r)]\colon r \leq z_0\}$.

(ii) Since $\chi_v\colon L_{a,v} \to \mathbb{R}$ is a bijection for every element a of A, (ii) follows from (i).

2.2. *A Complex of Åberg's Type*

Let V be a finite set of characters of Q with value groups, the group of rational integers $\mathbb{Z}$, and for every character v in V let X_v be the tree defined in the previous section with the additional property that all β_v, needed in the definition of G_v, are equal to a negative integer β. Later we will put some more restrictions on an upper bound of β in order to prove that the complex of Åberg's type constructed below has all the properties required in the Introduction.

We define a space $X = \prod_{v \in V} X_v$ and a map f: $X \to \prod_{v \in V} \mathbb{R}_v$ (where $\mathbb{R}_v = \mathbb{R}$ for every $v \in V$) given by $f(\prod_{v \in V}[(a_v, r_v)]) = \prod_{v \in V} r_v$ where a_v lies in A and r_v is a real number. Let χ: $Q \otimes \mathbb{R} \to \prod_{v \in V} \mathbb{R}_v$ be the $\mathbb{R}$-linear map extending the map sending an element q of Q to $\prod_{v \in V} v(q)$. Denote Y the subspace $\{x \in X\colon f(x) \in \operatorname{Im} \chi\}$ of X. If all X_v are valuation trees this construction is due to Åberg [1]. We make G act on X via the diagonal action (i.e., $g * \prod_{v \in V}[(a_v, r_v)] = \prod_{v \in V} g * [(a_v, r_v)]$).

LEMMA 2.2. *The space Y is invariant under this G-action.*

Proof. Let $\prod_{v \in V}[(a_v, r_v)]$ be a point of Y and let g be an element of G. Then $g * \prod_v [(a_v, r_v)] = \prod[(a'_v, r_v + \alpha_v(g))]$ where $g a_v \pi(q_v)^{[[r_v]]} G_v = a'_v \pi(q_v)^{[[r_v]] + \alpha_v(g)} G_v$ for some element a'_v of A and where $[[r_v]]$ is the rational integer such that $[[r_v]] - r_v \in [\,0, 1)$. We remind the reader that α_v: $G \to \mathbb{Z}$ is the group homomorphism extending the character v. Since $\prod_v \alpha_v(g) \in \operatorname{Im} \chi$ we see that Y is invariant under G-action.

All the constructions considered up to now are quite general but from now on we assume that $M := \mathbb{Z}Q/\operatorname{ann}_{\mathbb{Z}Q} A$ is a $\mathbb{Z}$-torsion ring of Krull dimension one. By [6] $\Sigma^c_M(Q) = \bigcup_P \Sigma^c_{M/P}(Q)$ where P runs through all minimal prime ideals of M and $\Sigma^c_{-}(Q)$ denotes the complement of $\Sigma_{-}(Q)$ in S^{n-1}. By [4] $\Sigma^c_{M/P}(Q)$ is the image of the projection to the unit sphere S^{n-1} in $\mathbb{R}^n$ of a rationally defined polyhedron of dimension that equals the Krull dimension of M/P. Since M/P has prime characteristic and Krull dimension at most one, $\Sigma^c_{M/P}(Q)$ is the image of the projection to S^{n-1} of finitely many rays in $\mathbb{R}^n$ and hence $\Sigma^c_{M/P}(Q)$ and $\Sigma^c_M(Q)$ are finite sets. Then there exists a cyclic subgroup Q' of Q such that $S(Q, Q') := \{\chi/\sim \mid \chi\colon Q \to \mathbb{R},\ \chi \neq 0,\ \chi(Q') = 0\}$ is contained in $\Sigma_M(Q)$ and, by [6, Corollary 4.5], M is finitely generated as a $\mathbb{Z}Q'$-module. We take q_0 to be a generator of Q' and define $y_1, \ldots, y_r$ to be generators of M as $\mathbb{Z}Q'$-module. Note that according to [5] $\Delta_A(Q) = \Delta_M(Q)$ and hence A is m-tame if and only if M is m-tame.

Our aim from now on is to construct a finite set of characters V and to choose the negative integer β from the definition of the complex Y such that:

(i) Y/G is a compact space;

(ii) Y is $(m-1)$-acyclic;

(iii) the stabilizer in G of a cell in Y is always polycyclic and hence of type FP_∞.

The construction of Y resembles that of Åberg's complex in [1]; in fact Lemma 2.1(ii) allows us to use Åberg's argument [1, Chap. 3, Proposition 3.3] verbatim to show that if the set of characters V is $1, 2, \ldots, (m-1)$-domesticated then Y is $(m-1)$-acyclic. We will construct the set V in such a way that all characters in V lie in $\Delta_M(Q)$ and by [1, Chap. 1, Proposition 6.3] if M is m-tame the set V is $1, 2, \ldots, (m-1)$-domesticated. This shows that Y is really $(m-1)$-acyclic.

We need to explain the structure of Y and X as CW-complexes. If the set V contains d characters then X is built by gluing d-cubes of the type $\{\prod_{v \in V}[(a_v, r_v)] : z_v \le r_v \le z_v + 1\}$ on some parts of their faces, where a_v are elements of A and z_v are rational integers. These d-cubes form the d-cells in X and every i-cell $(i \le d)$ of X is an i-cube of the face of a d-cell in X. The intersection of Y with an i-cell in X contained in the d-cube $\{\prod_v[(a_v, r_v)] : z_v \le r_v \le z_v + 1\}$ is homeomorphic to the intersection of the subspace im χ of $\prod_v \mathbb{R}_v$ with an i-cube contained in $\{\prod_v r_v \in \prod_v \mathbb{R}_v : z_v \le r_v \le z_v + 1\}$ and so is empty or homeomorphic to the unit j-cube B^j for some $j \le i$. Now we define a j-cell of Y to be every intersection of Y with a cell in X which is homeomorphic to B^j.

2.3. *The Construction of the Set of Characters V*

Let w_i $(i = 1, 2)$ be a character of Q' given by $w_i(q_0) = (-1)^{i+1}$ and let W_i $(i = 1, 2)$ be the set of all characters v of Q extending w_i such that M is not finitely generated as $\mathbb{Z}Q_v$-module. The last condition is equivalent to the graph X_v not being a single line and also equivalent to $v \in \Delta_M(Q) = \bigcup_j \Delta_{M/P_j}(Q)$ for a complete set of minimal prime ideals $P_1, \ldots, P_s$ in M. We claim that all the elements of W_i are discrete characters and W_i is a finite set for $i = 1, 2$.

Let $v \in \Delta_{M/P}(Q) \cap W_i$ for some minimal prime ideal P in M. Since M is finitely generated as $\mathbb{Z}[q_0^{\pm 1}]$-module, for any element q of Q there is a relation $f_0(q_0)q^\alpha + f_1(q_0)q^{\alpha-1} + \cdots + f_\alpha(q_0) = 0$ in M/P for some positive integer α and elements $f_i(q_0)$ of $\mathbb{Z}_p[q_0^{\pm 1}]$, where p is the characteristic of the ring M/P and $\mathbb{Z}_p$ denotes the finite field consisting of p elements. As shown in [4, Lemma 3.2] there is a substitution $q_0 = aq^s$ for an integer s such that the above relation becomes of the type $g_0(a)q^\beta + g_1(a)q^{\beta-1} + \cdots + g_\beta(a) = 0$ for some positive integer β and monomials $g_i(a) = z_i a^{\beta_i}$, $z_i \in \mathbb{Z}_p$, $0 \le i \le \beta$. Since M/P is not finitely generated as $\mathbb{Z}Q_v$-module, by [5, Proposition 2.1] $v(a^{\beta_i}q^{\beta-i}) = v(a^{\beta_j}q^{\beta-j})$ for some $i < j$ such that the coefficients z_i and z_j are non-trivial. Then $v(q) = (\beta_j - \beta_i)v(q_0)/((j-i) + s(\beta_j - \beta_i)) \in \mathbb{Q} \subset \mathbb{R}$; note that the denominator cannot be trivial otherwise $\beta_j - \beta_i = 0$ and hence $i = j$, a contradiction. Therefore $v(q) \in \mathbb{Q}$ for every element q of Q, in particular for elements of any finite generating set of Q. Thus the set W_i contains only discrete characters and is finite for each $i = 1, 2$.

Denote by V_i, for $i = 1, 2$, the set $\{d_v v : v \in W_i, d_v$ is a positive real number such that the image of $d_v v$ is exactly $\mathbb{Z}\}$ and let $V = V_1 \cup V_2$. We note that $\Sigma_A^c(Q) = \Sigma_M^c(Q) = V/\sim$.

2.4. *G Acts Cocompactly on Y*

A typical point of Y is $\prod_{v \in V}[(a_v, r_v)]$ where the a_v are elements of A and $\prod_v r_v \in \mathrm{Im}\ \chi \subseteq \prod_v \mathbb{R}_v$. In Theorem 2.4 we shall prove that for any such point there exists an element a of A such that $\prod_v[(a_v, r_v)] = \prod_v[(a, r_v)]$. Since $(-a) * \prod_v[(a, r_v)] = \prod_v[(0, r_v)]$ we have $Y/A \simeq \mathrm{Im}\ \chi$ and so $Y/G \simeq \mathrm{Im}\ \chi/Q$, where Q acts on $\prod_v \mathbb{R}_v$ via the map χ and $Q \simeq \mathbb{Z}^n$ acts additively on $\mathbb{R}^n$. The map from $\mathbb{R}^n/\mathbb{Z}^n \simeq (Q \otimes \mathbb{R})/Q$ to $\mathrm{Im}\ \chi/Q$ induced by χ is surjective and continuous and hence the space $\mathrm{Im}\ \chi/Q \simeq Y/G$, being an image of a compact space, is also compact.

Definition. If B is a subset of $\prod_v \mathbb{R}_v$ we define $[[B]] = \{\prod_v c_v \in \prod_v \mathbb{R}_v$: there exists $b = \prod_v b_v \in B$ such that $c_v - b_v \in [\,0, 1)$ and $c_v \in \mathbb{Z}$ for all $v \in V\}$.

Now we consider the case $B = \mathrm{Im}\ \chi$.

Lemma 2.3. $[[\mathrm{Im}\ \chi]]$ *is finite modulo the* Q*-action.*

Proof. Let $\prod_v b_v \in \mathrm{Im}\ \chi$, $[[\prod_v b_v]] = \prod_v c_v$, and let q be an element of Q. Then $[[q * \prod_v b_v]] = [[\prod_v(v(q) + b_v)]] = q * \prod_v c_v$ because $\prod_v v(q)$ is an integral point of $\prod_v \mathbb{R}_v$ and hence $[[\mathrm{Im}\ \chi]]$ is closed under the Q-action.

Let d be the number of the valuations in V. Since $[[\mathrm{Im}\ \chi]]$ consists of integral points $\prod_{v \in V} c_v$ and for every such point there exists an element $\prod_{v \in V} b_v$ of $\mathrm{Im}\ \chi$ such that $0 \le c_v - b_v < 1$ we have

$$[[\mathrm{Im}\ \chi]] \subseteq \bigcup_{r \in \Delta} ((r + \mathrm{Im}\ \chi) \cap \mathbb{Z}^d),$$

where $\Delta = \{r \in (\mathrm{Im}\ \chi)^\perp : |r| \le \sqrt{d}\}$. Since for every r of $\prod_v \mathbb{R}_v = \mathbb{R}^d$ the space $(r + \mathrm{Im}\ \chi)/Q$ is compact and $\mathbb{Z}^d/Q$ forms a discrete subset in $\mathbb{R}^d/Q$ we have $((r + \mathrm{Im}\ \chi) \cap \mathbb{Z}^d)/Q$ is finite. In order to prove Lemma 2.3 it is sufficient to show that the set $D = \{r \in \Delta : (r + \mathrm{Im}\ \chi) \cap \mathbb{Z}^d \neq \emptyset\}$ is finite. After an $\mathbb{R}$-linear transformation of the coordinate system we can assume that

$$\mathrm{Im}\ \chi = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{t \text{ times}} \times 0 \times \cdots \times 0,$$

$$(\mathrm{Im}\ \chi)^\perp = 0 \times \cdots \times 0 \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d - t \text{ times}}$$

for some positive integer t. Denote by L the image of the integral points in $\mathbb{R}^d$ under that transformation. Then L is a lattice in $\mathbb{R}^d$ and

$$D \subseteq \left\{ r \in (\mathrm{Im}\ \chi)^\perp : |r| \le c,\ (r + \mathrm{Im}\ \chi) \cap L \neq \varnothing \right\}$$

for some constant c. Since all the characters of V are discrete, L has a $\mathbb{Z}$-basis of rational vectors and so $L \subseteq (\mathbb{Z}/b)^d$ for some positive integer b. Then D is contained in the discrete set

$$\left\{ r \in (\mathrm{Im}\ \chi)^\perp : (r + \mathrm{Im}\ \chi) \cap (\mathbb{Z}/b)^d \neq \varnothing \right\}$$
$$= 0 \times \cdots \times 0 \times \underbrace{\mathbb{Z}/b \times \cdots \times \mathbb{Z}/b}_{d-t \text{ times}}.$$

Since D is bounded we obtain that D is finite as required.

THEOREM 2.4. *For every* $\prod_v s_v$ *in* $[[\mathrm{Im}\ \chi]]$ *and every set* $\{a_v \in A\colon v \in V\}$ *there exists an element a in A such that* $\prod_v[(a_v, s_v)] = \prod_v[(a, s_v)]$. *Consequently for every* $\prod_v r_v \in \mathrm{Im}\ \chi$ *such that* $[[\prod_v r_v]] = \prod_v s_v$ *one has* $\prod_v[(a_v, r_v)] = \prod_v[(a, r_v)]$.

In order to prove Theorem 2.4 we need the following proposition.

PROPOSITION 2.5. *Let B be a finitely generated $\mathbb{Z}Q$-module generated by the elements $b_1, \dots, b_k$ with annihilator I in $\mathbb{Z}Q$ such that $\mathbb{Z}Q/I$ is a ring of non-zero characteristic and Krull dimension one. Let $\{v_1, \dots, v_r\}$ be a set of discrete characters of Q lying in an open half subspace of* $\mathbb{R}^n = \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$ *with all $\mathbb{R}^+ v_i$ representing different rays in $\mathbb{R}^n$. Let B_{v_i} be the $\mathbb{Z}Q_{v_i}$-submodule of B generated by $b_1, \dots, b_k$ and let q_{v_i} be an element of Q such that $v_i(q_{v_i}) = 1$ for $1 \le i \le r$. Then for any set $\{b_{v_i}\colon 1 \le i \le r\}$ of elements of B and any set of integers $\{s_{v_i}\colon 1 \le i \le r\}$ there exists an element b in B such that $b - b_{v_i} \in q_{v_i}^{s_{v_i}} B_{v_i}$ for all i.*

Deduction of Theorem 2.4 *from Proposition* 2.5. By Lemma 2.3 $[[\mathrm{Im}\ \chi]]$ is finite modulo the action of Q. Let $\prod_v s_v = T_i + \prod_v v(q)$ for an element q of Q and a point $T_i = \prod_v r_{v,i} \in \prod_v \mathbb{R}_v$ from a fixed representative set $T_1, \dots, T_m$ of $[[\mathrm{Im}\ \chi]]$ modulo Q. We have

$$\prod_v [(a_v, s_v)] = \prod_v [(a, s_v)] \qquad \text{for some } a \in A$$

if and only if

$$a_v \pi(q_v)^{s_v} G_v = a \pi(q_v)^{s_v} G_v \qquad \text{for each } v \in V,$$

and this, in turn, is equivalent to

$$q_v^{-s_v}(a - a_v) \in G_v \cap A = q_v^{\beta} A_v \qquad \text{for each } v \in V.$$

Since $s_v + \beta = v(q) + r_{v,i} + \beta \leq v(q) + \beta_1$ where $\beta_1 = \max\{r_{v,i}: 1 \leq i \leq m, v \in V\} + \beta$, it is sufficient to find $a' \in A$ such that $a' - a'_v \in q_v^{\beta_1} A_v$ for every $v \in V$ where $a'_v = q^{-1} a_v \in A$. Then $a = qa'$ is an element with the required properties.

Since $A = Ma_1 + \cdots + Ma_s$ where $M = \mathbb{Z}Q/I$ and $I = \operatorname{ann}_{\mathbb{Z}Q} A$, it suffices to consider the case $s = 1$ and $a'_v = 0$ for $v \neq w$ and $a'_w = ma_1$ for some element m of M and a character w of V. Thus we reduce the problem to finding m' in M such that $m' \in q_v^{\beta_1}\mathbb{Z}Q_v + I/I$ for $v \neq w$ and $m' - m \in q_w^{\beta_1}\mathbb{Z}Q_w + I/I$, and then $a' = m'a_1$ is an element with the required properties.

Without loss of generality we can assume that w is a character from V_1 (see the end of Section 2.3 for the definition of V_1) and by Proposition 2.5 there exists an element $x \in M$ such that $x - m \in q_w^{\beta_1}\mathbb{Z}Q_w + I/I$ and $x \in q_v^{\beta_1}\mathbb{Z}Q_v + I/I$ for all $v \neq w$ in V_1. As shown in Section 2.2 we have that M is generated by $y_1, \ldots, y_r$ as $\mathbb{Z}[q_0^{\pm 1}]$-module. From the very beginning we can assume that β is a sufficiently small negative integer such that $y_i \in q_v^{\beta_1}\mathbb{Z}Q_v + I/I$ for all $v \in V$ and $1 \leq i \leq r$. Then $x = \sum_{i=1}^r (f_i + g_i) y_i$ for some $f_i \in \mathbb{Z}[q_0]$ and $g_i \in \mathbb{Z}[g_0^{-1}]$. We take $m' = x - \sum_i f_i y_i = \sum_i g_i y_i$ and hence $m' - x \in q_v^{\beta_1}\mathbb{Z}Q_v + I/I$ for $v \in V_1$ and $m' \in q_v^{\beta_1}\mathbb{Z}Q_v + I/I$ for $v \in V_2$. By the choice of x we obtain that m' has the desired property.

Proof of Proposition 2.5. 1. Let $I_1, \ldots, I_k$ be ideals of $\mathbb{Z}Q$ containing I and suppose that the proposition holds whenever $I_i \subseteq \operatorname{ann}_{\mathbb{Z}Q} B$. We show first that if there exists a chain

$$B = B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots \supseteq B_m \supseteq B_{m+1} = 0$$

of $\mathbb{Z}Q$-modules such that $\operatorname{ann}_{\mathbb{Z}Q}(B_i/B_{i+1}) \supseteq I_{s(i)}$ for every i and some integers $1 \leq s(i) \leq k$ then the proposition holds.

Let $\{b_{i,1}, \ldots, b_{i,m_i}\}$ be a set of generators of B_i as $\mathbb{Z}Q$-module. For a sufficiently large positive integer u we have $q_0^u b_{i,j} \in B_{v_t}$ for any $1 \leq t \leq r$, $1 \leq j \leq m_i$, $0 \leq i \leq m$, where q_0 is an element of Q such that $v_t(q_0) > 0$ for all t. Since the proposition holds for B_i/B_{i+1} generated by $\{q_0^u b_{i,j} + B_{i+1}: 1 \leq j \leq m_i\}$ as $\mathbb{Z}Q$-module we have that for any set of integers $\{s_t: 1 \leq t \leq r\}$ and any set $\{a_{i,t}: 1 \leq t \leq r\}$ of elements of B_i there exists an element λ_i in B_i such that

$$\lambda_i - a_{i,t} \in q_{v_t}^{s_t}\left(\sum_{j=1}^{m_i} \mathbb{Z}Q_{v_t} q_0^u b_{i,j}\right) + B_{i+1} \subseteq q_{v_t}^{s_t} B_{v_t} + B_{i+1}.$$

Using this property we find elements c_i in B_i such that $c_0 - b_{v_t} = d_{t,1} + c_{t,1}$, $c_i + c_{t,i} = d_{t,i+1} + c_{t,i+1}$ where $c_{t,i} \in B_i$, $d_{t,i} \in q_{v_t}^{s_t} B_{v_t}$ for $1 \leq i \leq m$, $1 \leq t \leq r$. Since $B_{m+1} = 0$ we have $c_{t,m+1} = 0$ and hence $b := c_0 + c_1 +$

$\cdots + c_m = b_{v_t} + d_{t,1} + \cdots + d_{t,m+1} \in b_{v_t} + q_{v_t}^{s_t} B_{v_t}$ for all t and therefore b has the required property.

2. In general $I \subseteq \sqrt{I} = P_1 \cap \cdots \cap P_s$ for some prime ideals $P_1, \dots, P_s$ in $\mathbb{Z}Q$ such that the Krull dimension of the rings $\mathbb{Z}Q/P_i$ is less than or equal to one. Let $\sqrt{I}^{m+1} \subseteq I$. Then

$$B \supseteq \sqrt{I} B \supseteq \sqrt{I}^2 B \supseteq \cdots \supseteq \sqrt{I}^{m+1} B = 0$$

and for the $\mathbb{Z}Q$-modules $B_i = \sqrt{I}^i B / \sqrt{I}^{i+1} B$ the following inclusions hold

$$B_i \supseteq P_1 B_i \supseteq (P_1 P_2) B_i \supseteq \cdots \supseteq (P_1 \cdots P_s) B_i = 0.$$

According to step 1 it is sufficient to consider the case $P \subseteq \operatorname{ann}_{\mathbb{Z}Q} B$ for some prime ideal P of $\mathbb{Z}Q$ such that the Krull dimension of $\mathbb{Z}Q/P$ is less than or equal to one.

In the case when the Krull dimension of $\mathbb{Z}Q/P$ is zero the ring $\mathbb{Z}Q/P$ is finite and hence B is finite. Then B is finitely generated as $\mathbb{Z}Q_v$-module and by [5, Proposition 2.1], $B = B_v = q_v^{s_v} B_v$ for any integer s_v and any real character v of Q. Then every b in B has the required property.

3. It remains to consider the case $P \subseteq \operatorname{ann}_{\mathbb{Z}Q} B$ where $\mathbb{Z}Q/P$ is an integral domain of Krull dimension one and non-zero characteristic. Then B is p-torsion for some prime integer p and $B = \mathbb{Z}_p Q b_1 + \cdots + \mathbb{Z}_p Q b_s$. With abuse of notation we write P for the image of the ideal P in $\mathbb{Z}_p Q$. To finish the proof of the proposition it suffices to consider the case $s = 1$ and $b_{v_i} = 0$ for $i \neq j$ and $b_{v_j} \in Q$ for some j between 1 and r. We aim to find an element λ in $\mathbb{Z}_p Q$ such that $\lambda \in q_{v_i}^{s_{v_i}} \mathbb{Z}_p Q_v + P$ for $i \neq j$ and $\lambda - b_{v_j} \in q_{v_j}^{s_{v_j}} \mathbb{Z}_p Q_{v_j} + P$. Now multiplying by $b_{v_j}^{-1}$ and changing the integers s_{v_i} to $s'_{v_i} = s_{v_i} - v_i(b_{v_j})$ we have to consider the above inclusions with $b_{v_j} = 1_Q$ and the new set of integers s'_{v_i}. The following lemma finishes off the proof of Proposition 2.5.

LEMMA 2.6. *In the previous notation for any set of integers* $\{s_{v_i} : 1 \le i \le r\}$ *there exists an element* λ *of* $\mathbb{Z}_p Q$ *such that* $\lambda \in q_{v_i}^{s_{v_i}} \mathbb{Z}_p Q_{v_i} + P$ *for* $1 \le i \neq j \le r$ *and* $\lambda - 1_Q \in q_{v_j}^{s_{v_j}} \mathbb{Z}_p Q_{v_j} + P$.

Proof. Since $\operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{Z}) \simeq \mathbb{Z}^n \simeq Q \subset \mathbb{R}^n$ we can consider the discrete characters $v_1, \dots, v_r$ as elements of the group $Q \subset \mathbb{R}^n$ and v_i as maps given by $v_i(q) = (v_i, q)$, where $(\ ,\ )$ is the standard inner product in $\mathbb{R}^n$. We prove Lemma 2.6 by induction on the rank of Q and assume for the moment that Lemma 2.6 holds for $r(Q) \le 2$. Let $n \ge 3$ be the rank of Q. We want to find a hyperplane L in $\mathbb{R}^n = Q \otimes \mathbb{R}$ defined by $(h, L) = 0$ for some rational vector h in $\mathbb{R}^n$ such that

(1) if $\pi: \mathbb{R}^n \to L$ is the projection map then all projections $\pi(v_i)$ are non-trivial;

(2) the rays in $\mathbb{R}^{n-1} \simeq L$ determined by $\pi(v_i)$ are all different and lie in an open half subspace of L.

Note that $\mathbb{R}\pi(v_i) = \mathbb{R}\pi(v_j)$ if and only if h, v_i, v_j are linearly dependent.

Let $h_1 = q_0 + \epsilon e$ where e is a unit vector in $\mathbb{R}^n$, ϵ a real number, and q_0 is an element of Q such that $v_i(q_0) > 0$ for all i. If $|\epsilon|$ is sufficiently small then $(v_i, h_1) > 0$ for all i. If in addition $(h, h_1) = 0$ then $(\pi(v_i), h_1) = (v_i, h_1) > 0$ for all i, and so all $\pi(v_i)$ lie in the open half subspace $\{l \in L\colon (l, h_1) > 0\}$ of L. Thus if $h \in \bigcup_{|h_1 - q_0| \le \epsilon} h_1^{\perp} \setminus \bigcup_{i \ne j} \operatorname{span}\{v_i, v_j\}$ and h is a rational vector in $\mathbb{R}^n$, the hyperplane L defined by h has the required properties. Since the dimension of any of the spaces $h_1^{\perp}$ is $n-1$, and thus at least 2, and the first union is infinite we have that $\bigcup_{|h_1 - q_0| \le \epsilon} h_1^{\perp} \setminus \bigcup_{i \ne j} \operatorname{span}\{v_i, v_j\}$ is non-empty and obviously the rational points are dense in it. Finally we obtain a hyperplane L with the required properties.

Now let w_i be integral vectors lying on the rays defined by $\pi(v_i)$ and we consider w_i as discrete characters of $Q_1 = L \cap \mathbb{Z}^n \simeq \mathbb{Z}^{n-1} \subset Q \simeq \mathbb{Z}^n$ defined by $w_i(q) = (w_i, q)$ for $q \in Q_1$. We apply induction for the group Q_1. Then there exists an element λ in $\mathbb{Z}_p Q_1 \subseteq \mathbb{Z}_p Q$ such that $\lambda \in q_{w_i}^{\alpha_i} \mathbb{Z}_p (Q_1)_{w_i} + (P \cap \mathbb{Z}_p Q_1)$ for $i \ne j$ and $\lambda - 1 \in q_{w_j}^{\alpha_j} \mathbb{Z}_p (Q_1)_{w_j} + (P \cap \mathbb{Z}_p Q_1)$ where q_{w_i} are elements of Q_1 such that $w_i(q_{w_i}) = 1$, α_i are rational integers, and $(Q_1)_{w_i} = \{q \in Q_1 \mid w_i(q) \ge 0\}$. Since $v_i(q_{w_i}) > 0$, we have $q_{w_i}^{\alpha_i} \mathbb{Z}_p (Q_1)_{w_i} \subseteq q_{v_i}^{s_{v_i}} \mathbb{Z}_p Q_{v_i}$ for a sufficiently big α_i and then λ is an element with the required properties.

Now we consider the case $n = 2$ (if $n = 1$ then r should be 1 as well so there is nothing to prove). We write x, y for the standard generators of $Q = \mathbb{Z}^2 \subset \mathbb{R}^2$ and without loss of generality we can assume that $v_i(y) > 0$ for all i. By reordering the set $\{v_1, v_2, v_3, \dots, v_{r-1}, v_r\}$ if necessary we can assume that $v_i(x)/v_i(y) < v_{i+1}(x)/v_{i+1}(y)$ for $1 \le i \le r-1$. Since $\mathbb{Z}_p Q_{v_i} \not\equiv \mathbb{Z}_p Q \bmod P$ if and only if v_i belongs to $\Delta_{\mathbb{Z}_p Q/P}(Q)$ (see [5, Proposition 2.1]) we can assume without loss of generality that all v_i have this property. By enlarging the set of characters $\{v_1, \dots, v_r\}$ we can assume as well that the classes of $v_1, \dots, v_r$ represent all the elements in $\Sigma^c_{\mathbb{Z}_p Q/P}(Q)$ lying in the open half subspace of $\mathbb{R}^2$ defined by y.

Since the Krull dimension of $\mathbb{Z}_p Q/P$ equals one there exists a polynomial $t \in \mathbb{Z}_p[x, y]$ such that $P = (t)$. Let M be the convex hull of the support of t in $\mathbb{R}^2 = \mathbb{R} \otimes_{\mathbb{Z}} Q$. We order clockwise the edges of the boundary of M consecutively $l_1, \dots, l_s$ and by [6] for every i between 1 and r there exists $\sigma(i)$ such that v_i is perpendicular to $l_{\sigma(i)}$ and for any $m \in M \cap Q$ and $q \in l_{\sigma(i)} \cap Q$ we have $v_i(m) \ge v_i(q)$. Using the description of the geometrical invariant $\Sigma_C(Q)$ in [6] for one related kQ-module

C where k is a commutative ring with 1, the field $\mathbb{Z}_p$ in our case, and reordering $l_1, \ldots, l_s$ by translating the indices modulo s if necessary, we can assume $\sigma(i) = i$ for $i \leq r$.

Let P_i be the intersection of the edges l_{i-1} and l_i in $\mathbb{R}^2$ where $l_0 = l_s$ and let $q_1, \ldots, q_s$ be the elements of $Q \simeq \mathbb{Z}^2$ corresponding to the points $P_1, \ldots, P_s$. Let $t_1 = c_1^{-1} q_j^{-1} t$ and $t_2 = c_2^{-1} q_{j+1}^{-1} t$ where each c_i is the coefficient of q_{j+i-1} in t. If v is a vector in $\mathbb{R}^2$ and $g = \sum_q z_q q$ is an element in $\mathbb{Z}_p Q$ we define $g_v = \sum_{(q,v) \geq 0} z_q q$ and $g_v^+ = \sum_{(q,v)>0} z_q q$. Denote $\lambda_1 = (t_2)_{x^{-1}} + (t_1)_x - 1_Q$. Now $\lambda_1 \equiv ((t_2)_{x^{-1}} - 1_Q) - (t_1)^+_{x^{-1}} \bmod P$ and the elements of the supports of $(t_1)^+_{x^{-1}}$ and $(t_2)_{x^{-1}} - 1_Q$ lie in the open half subspaces defined by v_i for $1 \leq i \leq j-1$. We have as well $\lambda_1 \equiv ((t_1)_x - 1_Q) - (t_2)^+_x \bmod P$ and the elements of the supports of $(t_1)_x - 1_Q$ and $(t_2)^+_x$ lie in the open half subspaces of $\mathbb{R}^2$ defined by v_i for $j+1 \leq i \leq r$. Thus $\lambda_1 \in \mathbb{Z}_p Q^+_{v_i} + P$ for $1 \leq i \neq j \leq r$ where $Q^+_{v_i} = \{q \in Q : v_i(q) > 0\}$.

By the construction of λ_1, the support of $\lambda_1 - 1_Q$ lies in the open half subspace of $\mathbb{R}^2$ defined by v_j so $\lambda_1 - 1_Q \in \mathbb{Z}_p Q^+_{v_j} + P$. Since $(\lambda_1 - 1_Q)^{p^k} = \lambda_1^{p^k} - 1_Q$, for a sufficiently large positive integer k we obtain that $\lambda = \lambda_1^{p^k}$ has the required properties. This finishes the proof of Lemma 2.6.

As a corollary of Theorem 2.4 we get

COROLLARY 2.7. *Y/G is a compact space.*

2.5. *Stabilizers in G of Cells in Y*

We aim to show that if P is the stabilizer of a vertex $\prod_{v \in V} g_v G_v$ in X lying in $f^{-1}([[\text{im}\ \chi]])$ then P is polycyclic and hence of type FP_∞. As we will see later the stabilizer in G of a cell in Y is always of the form described above.

PROPOSITION 2.8. *The intersection $P \cap A$ is a finite group.*

Proof. If a is an element of A stabilizing a vertex $g_v G_v$ of X_v where $g_v = b\pi(q_v)^z$ for some element b of A and some rational integer z, we have $(b\pi(q_v)^z)^{-1} a b \pi(q_v)^z = q_v^{-z} a \in G_v \cap A = q_v^{\beta} A_v$ and so $a \in q_v^{\beta + \alpha_v(g_v)} A_v$. Then $P \cap A \subseteq \bigcap_{v \in V} q_v^{\alpha} A_v$ for some sufficiently small integer α.

We claim that if B_v are finitely generated $\mathbb{Z}Q_v$-submodules of A then the intersection $\bigcap_{v \in V} B_v$ is always finite. Then Proposition 2.8 obviously holds. We divide the proof into several steps.

(1) Let $I = \operatorname{ann}_{\mathbb{Z}Q} A$ and let $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_j = \mathbb{Z}Q$ be an increasing sequence of ideals in $\mathbb{Z}Q$ such that $I_0 \subseteq I$. Then $((\bigcap_v B_v) \cap I_i A)/((\bigcap_v B_v) \cap I_{i-1} A)$ embeds in $\bigcap_v ((B_v \cap I_i A) + I_{i-1}A/I_{i-1}A) \subseteq I_i A/I_{i-1}A$. We assume the claim holds for the $\mathbb{Z}Q$-modules $I_i A/I_{i-1}A$ for $1 \leq i \leq j$. Since the ring $\mathbb{Z}Q_v$ is Noetherian for a discrete character v, we have that $B_v \cap I_i A$ is finitely generated as $\mathbb{Z}Q_v$-module and therefore the intersection $\bigcap_v ((B_v \cap I_i A) + I_{i-1}A/I_{i-1}A)$ is finite for $1 \leq i \leq j$. Then all the quotients $((\bigcap_v B_v) \cap I_i A)/((\bigcap_v B_v) \cap I_{i-1}A)$ for $1 \leq i \leq j$ are finite and then $\bigcap_v B_v$ is finite as required.

(2) Let the radical $\sqrt{I}$ of I be the intersection of some prime ideals $P_1, \ldots, P_s$ in $\mathbb{Z}Q$ and $\sqrt{I}^k \subseteq I$ for some positive integer k. Then the Krull dimension of the rings $\mathbb{Z}Q/P_i$ for $1 \leq i \leq s$ is less than or equal to 1. The sequence of ideals in $\mathbb{Z}Q$, $\sqrt{I}^k \subseteq \sqrt{I}^{k-1} \subseteq \cdots \subseteq \sqrt{I} \subseteq \mathbb{Z}Q$, and the above remark allow us to assume that $P_1 \cap \cdots \cap P_s \subseteq \operatorname{ann}_{\mathbb{Z}Q} A$. In this case the following sequence of ideals in $\mathbb{Z}Q$, $P_1 \cap \cdots \cap P_s \subseteq P_1 \cap \cdots \cap P_{s-1} \subseteq \cdots \subseteq P_1 \subseteq \mathbb{Z}Q$, shows that we can further assume that $\operatorname{ann}_{\mathbb{Z}Q} A$ contains a prime ideal P of $\mathbb{Z}Q$ such that the Krull dimension of $\mathbb{Z}Q/P$ is less than or equal to one.

(3) Let $A = \mathbb{Z}Qa_1 + \cdots + \mathbb{Z}Qa_s$ and $P \subseteq \operatorname{ann}_{\mathbb{Z}Q} A$ as above. We use induction on s to show that if B_v are finitely generated $\mathbb{Z}Q_v$-submodules of A then the intersection $\bigcap_{v \in V} B_v$ is finite.

(3.1) We assume $s > 1$ and the statement is true for $\mathbb{Z}Q/P$-modules generated by at most $s-1$ elements. Let $J = \{\lambda \in \mathbb{Z}Q : \lambda a_1 \in \mathbb{Z}Qa_2 + \cdots + \mathbb{Z}Qa_s\}$. If P is strictly contained in J then either the radical $\sqrt{J}$ of the ideal J is an intersection of finitely many prime ideals $Q_1, \ldots, Q_t$ in $\mathbb{Z}Q$ strictly containing P or $\sqrt{J} = \mathbb{Z}Q$. In the first case $Q_1, \ldots, Q_t$ are maximal ideals of $\mathbb{Z}Q$ and hence they are cofinite in $\mathbb{Z}Q$. In both cases $\mathbb{Z}Q/J$ is a finite ring and so $A/\mathbb{Z}Qa_2 + \cdots + \mathbb{Z}Qa_s$ is finite. Then $(\bigcap_{v \in V} B_v)/((\mathbb{Z}Qa_2 + \cdots + \mathbb{Z}Qa_s) \cap (\bigcap_v B_v))$ is finite and by induction $(\bigcap_{v \in V} B_v) \cap (\mathbb{Z}Qa_2 + \cdots + \mathbb{Z}Qa_s)$ is finite again. We obtain that $\bigcap_{v \in V} B_v$ is finite as required.

In the case when $P = J$, $A = \mathbb{Z}Qa_1 \oplus (\mathbb{Z}Qa_2 + \cdots + \mathbb{Z}Qa_s)$. Let $\{b_1, b_2, \ldots, b_t\}$ be a finite set containing a generating set of B_v as $\mathbb{Z}Q_v$-module for all $v \in V$, so $B_v \subseteq \sum_{i=1}^t \mathbb{Z}Q_v b_i$. We write $b_i = b_{i,1} + b_{i,2}$ where $b_{i,1} \in \mathbb{Z}Qa_1$ and $b_{i,2} \in \mathbb{Z}Qa_2 + \cdots + \mathbb{Z}Qa_s$. Thus $\bigcap_v B_v \subseteq (\bigcap_v B_{v,1}) \oplus (\bigcap_v B_{v,2})$ where $B_{v,j} = \sum_i \mathbb{Z}Q_v b_{i,j}$ for $j = 1, 2$ and $v \in V$. By induction both $\bigcap_v B_{v,1}$ and $\bigcap_v B_{v,2}$ are finite and hence $\bigcap_v B_v$ is finite.

(3.2) It remains to consider the case $s = 1$, $A \simeq \mathbb{Z}Q/I$ for an ideal I in $\mathbb{Z}Q$ containing P where $\mathbb{Z}Q/P$ is an integral domain of non-zero characteristic and Krull dimension 1. If $P \neq I$ then A is finite and there is nothing to prove. If $I = P$ then $B_v \subseteq q_v^r \mathbb{Z}Q_v + I/I$ for a sufficiently small negative integer r and every $v \in V$ where q_v is an element of Q such that $v(q_v) = 1$. We remind the reader that the ring $M = \mathbb{Z}Q/\mathrm{ann}_{\mathbb{Z}Q} A$ for the initial $\mathbb{Z}Q$-module A is finitely generated as $\mathbb{Z}[q_0^{\pm 1}]$-module (see Sect. 2.2) and hence the new ring A is finite over the subring $S = \mathbb{Z}[q_0^{\pm 1}] + I/I$. Since A is infinite $S = \mathbb{Z}[q_0^{\pm 1}] + I/I$ is isomorphic to the Laurent polynomial ring $\mathbb{Z}_p[q_0^{\pm 1}]$, where p is the characteristic of the integral domain $A = \mathbb{Z}Q/I$.

Let V_i' $(i = 1, 2)$ be the set of all valuations of the field of fractions of A with the group of rational integers $\mathbb{Z}$ as value group with the additional property that the restriction of every element of V_i' to the Laurent polynomial ring $S \simeq \mathbb{Z}_p[q_0^{\pm 1}]$ is given by minimal formula and the value of q_0 is positive for $i = 1$ and negative for $i = 2$. A valuation v of $\mathbb{Z}_p[q_0^{\pm 1}]$ is given by minimal formula if $v(\sum_{k \in \mathbb{Z}} z_k q_0^k) = \min\{v(q_0^k)\colon z_k \neq 0\}$ where $z_k \in \mathbb{Z}_p$. Then the restrictions of the valuations in V_i' to Q give some elements of V_i. We set $V' = V_1' \cup V_2'$.

Now

$$\bigcap_{v \in V} B_v \subseteq \{a \in A | v(a) \geq r,\ v \in V'\} =: C'.$$

Since $v(q_0) \geq 1$ for every $v \in V_1'$ we have

$$q_0^{-r} C' \subseteq C_1 := \{a \in A \mid v(a) \geq 0 \text{ if } v \in V_1',\ v(a) \geq c \text{ if } v \in V_2'\}$$

for a negative integer $c \leq r(1 - \min\{v(q_0)\colon v \in V_2'\})$. Thus $C_1 \subseteq \{a \in A\colon v(a) \geq 0,\ v \in V_1'\}$ and by [8, Chap. 6.1.3, Theorem 3] $\{a \in A\colon v(a) \geq 0,\ v \in V_1'\} = \{a \in A : a$ is integral over $O_{v_1}\}$ where $O_{v_1} = \{k \in$ the field of fractions of $S\colon v_1(k) \geq 0\}$, v_1 is the valuation on the field of fractions of S given by minimal formula on S, and $v_1(q_0) = 1$. Since A is integral over $S = \mathbb{Z}_p[q_0^{\pm 1}]$ and O_{v_1} is a principal ideal domain, and so a unique factorization domain, any element of C_1 is integral over $S \cap O_{v_1} = \mathbb{Z}_p[q_0] =: E$. Thus $C_1 \subseteq D$ where D is the integral closure of E in the field of fractions K of A. By [8, Chap. 5.3.2, Theorem 2], D is a finitely generated E-module, say $D = Ed_1 + \cdots + Ed_m$. We define

$$D_k' = \left\{ e_1 d_1 + \cdots + e_m d_m \in D \,\middle|\, e_i = \sum_{0 \leq \alpha < k} z_{\alpha,i} q_0^{\alpha},\ z_{\alpha,i} \in \mathbb{Z}_p \right\},$$

$$D_k'' = \left\{ e_1 d_1 + \cdots + e_m d_m \in D \,\middle|\, e_i = \sum_{\alpha \geq k} z_{\alpha,i} q_0^{\alpha},\ z_{\alpha,i} \in \mathbb{Z}_p \right\}.$$

We show that for sufficiently large k, both $C_1 \cap D_k''$ and $C_1/(C_1 \cap D_k'')$ are finitely generated $\mathbb{Z}_p$-modules and consequently $C_1, C', \bigcap_{v \in V} B_v$, and $A \cap P$ are finite.

Generally D/D''_k is a finitely generated $\mathbb{Z}_p$-module so we need to show that $C_1 \cap D''_k$ is finitely generated for sufficiently large k. Let $l \in C_1 \cap D''_k$ and so $l = q_0^k l_1$ for some $l_1 \in D$ and $v(l_1) = v(l) - kv(q_0) \geq c - kv(q_0) \geq 0$ for sufficiently large k and for all v in V'_2. Thus $C_1 \cap D''_k \subseteq q_0^k\{m \in D\colon v(m) \geq 0,\ v \in V'_2\} = q_0^k\{$the integral closure of $\mathbb{Z}_p$ in $K\}$. The integral closure of $\mathbb{Z}_p$ in K is finitely generated as $\mathbb{Z}_p$-module by [8, Chap. 5.3.2, Theorem 2] because the algebraic closure of $\mathbb{Z}_p$ in K is a finite extension of $\mathbb{Z}_p$. This completes the proof of Proposition 2.8.

LEMMA 2.9. *If Γ is a cell of Y then the stabilizer of Γ in G coincides with the stabilizer in G of a point of $f^{-1}([[\mathrm{Im}\ \chi]])$.*

Proof. Let Γ be the intersection of an i-subcell of the d-cell $J = \{\prod_v[(a_v, r_v)]\colon z_v \leq r_v \leq z_v + 1\}$ in X with Y, let $\gamma_1, \ldots, \gamma_s$ be the vertices (0-subcells) of Γ, and let $g \in G$ stabilize the cell Γ. Then g permutes the vertices $\gamma_1, \ldots, \gamma_s$, say $g * \gamma_i = \gamma_{\rho(i)}$ for some permutation ρ. If $\gamma_i = \prod_v[(a_v, s_{v,i})]$ we have $s_{v,i} + \alpha_v(g) = s_{v,\rho(i)}$ and so $\sum_i(s_{v,i} + \alpha_v(g)) = \sum_i s_{v,\rho(i)} = \sum_i s_{v,i}$. Hence $\alpha_v(g) = 0$ for $v \in V$ and $\gamma_{\rho(i)} = \gamma_i$ for all i. Thus g stabilizes the vertices $\gamma_1, \ldots, \gamma_s$ and hence stabilizes Γ pointwise.

We note that the element g in G stabilizes the point $\prod_v[(a_v, r_v)]$ of X if and only if g stabilizes the vertex $\prod_v[(a_v, s_v)]$ of X where $\prod_v s_v = [[\prod_v r_v]]$, i.e., $s_v \in \mathbb{Z}$ and $s_v - r_v \in [0,1)$ for all v. We obtain that $\{g \in G : g$ stabilizes Γ pointwise$\} = \{g \in G : g$ stabilizes all points from the set $\Lambda\}$ where $\Lambda = \{\prod_v[(a_v, s_v)]\colon$ there exists a point $\prod_v[(a_v, r_v)]$ in Γ such that $\prod_v s_v = [[\prod_v r_v]]\} \subseteq$ vertices in X of the d-cell J. Let $\Lambda = \{T_1, \ldots, T_r\}$ and $f(T_i) = [[f(S_i)]]$ for some points $S_1, \ldots, S_r$ of Γ and $T_j = \prod_v[(a_v, z_{v,j})]$ for $1 \leq j \leq r$. We claim that the point $T = \prod_v[(a_v, z_v)]$ where $z_v = \max\{z_{v,j}\colon 1 \leq j \leq r\}$ belongs to Λ. Indeed $(1/r)(S_1 + \cdots + S_r)$ is a point of Γ and $[[f((1/r)(S_1 + \cdots + S_r))]] = f(T)$. Thus $\{g \in G : g$ stabilizes Γ pointwise$\} = \{g \in G : g$ stabilizes the vertex T of $X\}$ and T is a point of $f^{-1}([[\mathrm{Im}\ \chi]])$.

3. PROOF OF THEOREM B

3.1. *Åberg's Construction*

From now on we work in the notation of Theorem B. Without loss of generality we can assume Q is free abelian of rank n. To prove Theorem B we follow Åberg's construction from [1].

Let $P_1, \ldots, P_s$ be all the minimal associated primes for the $\mathbb{Z}Q$-module A and so $P_i = \mathrm{ann}_{\mathbb{Z}Q}(a_i)$ for some elements $a_1, \ldots, a_s$ of A. There is a $\mathbb{Z}Q$-module homomorphism $i\colon N = \mathbb{Z}Q/P_1 \oplus \cdots \oplus \mathbb{Z}Q/P_s \to A$ which sends $\lambda_1 \oplus \cdots \oplus \lambda_s$ to $\lambda_1 a_1 + \cdots + \lambda_s a_s$. By [1, Chap. 2, Proposition 1.4], i is an injection. From now on we consider N as a submodule of A.

We assume A is not m-tame but it is $(m-1)$-tame, and, as in the introduction of [11], there exist m discrete characters $\chi_1, \chi_2, \dots, \chi_m$ of Q each with value group a subgroup of the group of rational integers $\mathbb{Z}$. Moreover for every $1 \le i \le m$ at least one of the rings $\mathbb{Z}Q/P_j$, $1 \le j \le s$, is not finitely generated as $\mathbb{Z}Q_{\chi_i}$-module, where $Q_{\chi_i} = \{q \in Q\colon \chi_i(q) \ge 0\}$, and the convex hull of $\chi_1, \chi_2, \dots, \chi_m$ in $\mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R}) \simeq \mathbb{R}^n$ is an $(m-1)$-dimensional simplex containing the origin as an internal point. We note that if $z_1, z_2, \dots, z_m$ are positive integers then the characters $z_1\chi_1, \dots, z_m\chi_m$ still have the described properties. By multiplying χ_m with a sufficiently big positive integer, if necessary, we can assume that χ_m is an integral combination of $\chi_1, \dots, \chi_{m-1}$.

We view Q as the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. Then there is an isomorphism $\theta\colon \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z}) \to \mathbb{Z}^n = Q$ given by $\chi(q) = (\theta(\chi), q)$ for all $q \in Q$ and each character χ of Q, where $(\ ,\ )$ is the standard inner product in $\mathbb{R}^n$. Denote the subgroup of Q generated by $\theta(\chi_1), \dots, \theta(\chi_m)$ by Q' and let Q'' be the subgroup of $Q = \mathbb{Z}^n$, consisting of all elements in Q perpendicular to Q' with respect to the standard inner product in $\mathbb{R}^n$. Then $Q' \times Q''$ is a subgroup of finite index in Q and since the FP_m-Conjecture holds for a group G if and only if it holds for a subgroup of finite index in G we can assume $Q = Q' \times Q''$.

Åberg notices in [1] that if G is of type FP_m the canonical map

$$H^0(Q', E) = E^{Q'} \to H_0(Q'', E) = E_{Q''}$$

is trivial where E stands for $(\mathbb{Z}G^{\mathbb{N}})_A$. We aim to construct an element of the Cartesian product $\mathbb{Z}G^{\mathbb{N}}$ whose images β in E is Q'-invariant and the image of β in $E_{Q''}$ is non-trivial. We need some properties of the elements of $\mathbb{Z}G^{\mathbb{N}}$ which vanish in $(\mathbb{Z}G^{\mathbb{N}})_{G''}$, where G'' is the subgroup of G containing A and $G''/A \simeq Q''$.

Let $\{F^d\}_{d\in\mathbb{Z}}$ be a decreasing filtration of A such that $\bigcup_{d\in\mathbb{Z}} F^d = A$. For an element f of the augmentation ideal of the group algebra $\mathbb{Z}A$ there is an order function $o(f)$ (see [1]) given by

$$o(f) = \sup\left\{d\colon f \equiv 0 \text{ in } \mathbb{Z}\left[A/F^d\right]\right\} \in \mathbb{Z} \cup +\infty.$$

The following proposition in the split case is proved in [1, 11].

Proposition 3.1. *Let $A \to G \to Q$ be a short exact sequence of groups with A, Q abelian, G finitely generated, and let $\pi\colon Q \to G$ be a lifting of the projection map $G \to Q$. Let $\{F^d\}_{d\in\mathbb{Z}}$ be a decreasing filtration of $\mathbb{Z}Q''$-submodules of A such that $\bigcup_{d\in\mathbb{Z}} F^d = A$ and let $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ be an element of $\mathbb{Z}G^{\mathbb{N}}$ with $\alpha_j = \sum_q \alpha_{j,q}\pi(q)$, $\alpha_{j,q} \in \mathbb{Z}A$, $q \in Q'$. If α vanishes in $(\mathbb{Z}G^{\mathbb{N}})_{G''}$, where G'' is the subgroup of G generated by A and $\pi(Q'')$, then $\alpha_{j,q}$ are elements of the augmentation ideal of $\mathbb{Z}A$ and* $\inf\{o(\alpha_{j,q})\colon j, q\} > -\infty$.

Remark. In the case when all F^d are finitely generated $\mathbb{Z}Q''$-modules and $\inf\{o(\alpha_{j,q})\colon j,k\} > -\infty$ it can be shown that α vanishes in $(\mathbb{Z}G^{\mathbb{N}})_{G''}$ (a split version is proved in [1, 11]) but as the filtration we will consider later consists of $\mathbb{Z}Q''$-modules which are not finitely generated, we abstain from claiming sufficiency.

Proof of Proposition 3.1. We follow the proof of [11, Chap. 2.3, Proposition]. Let α vanish in $(\mathbb{Z}G^{\mathbb{N}})_{G''}$ and $\alpha = \sum_{i=1}^{s}(g_i'' - 1)\beta_i$ for some $g_i'' \in G''$, $\beta_i \in \mathbb{Z}G^{\mathbb{N}}$. Let $\beta_i = (\beta_{i,j})_{j\in\mathbb{N}}$ where $\beta_{i,j} = \sum_q \beta_{i,j,q}\pi(q)$ and $\beta_{i,j,q} \in \mathbb{Z}G''$, $q \in Q'$. Then $\alpha_{j,q} = \sum_{i=1}^{s}(g_i'' - 1)\beta_{i,j,q}$ for any positive number j and an element q of Q'. We write $g_i'' = a_i\pi(q_i'')$ for some $a_i \in A$ and $q_i'' \in Q''$, and define P to be the subgroup of G'' generated by $\pi(Q'')$ and $\{a_i\colon 1 \le i \le s\}$. Then $\alpha_{j,q}$ vanishes in $\mathbb{Z}[G''/P] \simeq \mathbb{Z}[A/(A\cap P)]$; here $\mathbb{Z}[G''/P]$ and $\mathbb{Z}[A/(A\cap P)]$ are the free abelian groups on the right coset classes of P in G'' and $A \cap P$ in A, respectively, and the isomorphism is of free abelian groups given by the obvious bijection between the coset classes. Since $P \cap A$ is finitely generated as $\mathbb{Z}Q''$-module it is contained in some F^d and so $\alpha_{j,q}$ vanishes in $\mathbb{Z}[A/F^d]$. Thus $\alpha_{j,q}$ are elements of the augmentation ideal of $\mathbb{Z}A$ and $o(\alpha_{j,q}) \ge d$ for all j, q. Later we will need the following lemma.

LEMMA 3.2. *Let A be a $\mathbb{Z}Q$-module, let R be a Noetherian subring of $\mathbb{Z}Q$, and let N be an R-submodule of A. If $\{F^d\}_{d\in\mathbb{Z}}$ is a decreasing filtration of R-submodules of N such that $\bigcup_d F^d = N$ then there exists a decreasing filtration $\{F_*^d\}_{d\in\mathbb{Z}}$ of R-submodules of A such that $F_*^d \cap N = F^d$ and $\bigcup_{d\in\mathbb{Z}} F_*^d = A$.*

Proof. Let

$$N = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\lambda \subseteq \cdots \subseteq N_\beta = A$$

be a series of R-submodules of A with each $N_{\lambda+1}/N_\lambda$ being a cyclic R-module and for any limit ordinal λ the R-module N_λ is the union of all R-modules N_μ for $\mu < \lambda$. We use transfinite induction on β to prove that there is a decreasing filtration $\{F_\mu^d\}_{d\in\mathbb{Z}}$ of R-submodules of N_μ for each $\mu \le \beta$ with the properties that $\bigcup_{d\in\mathbb{Z}} F_\mu^d = N_\mu$ for all $\mu \le \beta$, if μ is not a limit ordinal $F_\mu^d \cap N_{\mu-1} = F_{\mu-1}^d$, and if μ is a limit ordinal $F_\mu^d = \bigcup_{\lambda<\mu} F_\lambda^d$.

First we consider the case when $\beta = 1$, so $A = N + Ra_0$ for some element a_0 of A.

Let $I = \{\lambda \in R\colon \lambda a_0 \in N\}$ and $\{U^d\}_{d\in\mathbb{Z}}$ be the decreasing filtration of R-submodules of I given by $U^d = \{\lambda \in I\colon \lambda a_0 \in F^d\}$. Since $I = \bigcup_d U^d$ and R is Noetherian $I = U^{d_0}$ for some integer d_0. We define a decreasing filtration $\{U_1^d\}_{d\in\mathbb{Z}}$ of R by $U_1^d = U^d$ for $d \ge d_0$ and $U_1^d = R$ for $d \le d_0 - 1$. This gives a decreasing filtration $\{F_1^d\}$ of A such that $a \in F_1^d$ if and only if $a = a' + \lambda a_0$ for some a' in $F^d \subseteq N$ and $\lambda \in U_1^d$. If in addition a is an element of N then λ is an element of $I = U^{d_0}$. In this situation if $d_0 \ge d$

then $\lambda a_0 \in U^{d_0} a_0 \subseteq F^{d_0} \subseteq F^d$ and if $d_0 \leq d$ the inclusion $U_1^d a_0 = U^d a_0 \subseteq F^d$ yields as well $\lambda a_0 \in F^d$. Thus $F_1^d \cap N = F^d$ as required.

For the inductive step suppose we have constructed decreasing filtrations $\{F_\mu^d\}_{d \in \mathbb{Z}}$ for all $\mu < \beta$. If β is a limit ordinal define $F_\beta^d = \bigcup_{\mu < \beta} F_\mu^d$ for all $d \in \mathbb{Z}$. In the case when β is not a limit ordinal we use the same as in the case $\beta = 1$ to construct a filtration $\{F_\beta^d\}_{d \in \mathbb{Z}}$ such that $F_\beta^d \cap N_{\beta - 1} = F_{\beta - 1}^d$ for all $d \in \mathbb{Z}$.

Finally we define F_*^d to be F_β^d for all $d \in \mathbb{Z}$.

3.2. *An Element of Åberg's Type*

We amend slightly the notation given in Section 3.1. We remind the reader that there are m characters $\chi_1, \ldots, \chi_m$ of Q such that $\chi_i(q) = (q_{\chi_i}, q)$ for all $q \in Q$ and some elements $q_{\chi_1}, \ldots, q_{\chi_m}$ in Q, where $Q = \mathbb{Z}^n \subset \mathbb{R}^n$ and $(\, , \,)$ is the standard inner product in $\mathbb{R}^n$. Furthermore, q_{χ_m} lies in the subgroup Q' of Q generated by $q_{\chi_1}, q_{\chi_2}, \ldots, q_{\chi_{m-1}}$ and $Q = Q' \times Q''$ where Q'' is the orthogonal complement of Q' in Q with respect to the standard inner product in $\mathbb{R}^n$.

For any character χ_i we define a filtration $\{F_{\chi_i}^d\}_{d \in \mathbb{Z}}$ of $\mathbb{Z}Q_{\chi_i}$-submodules of $N = \mathbb{Z}Q/P_1 \oplus \cdots \oplus \mathbb{Z}Q/P_s$ as follows. Let $A_j = \mathbb{Z}Q/P_j$, $A_{j, \chi_i}^d = (q_{\chi_i}^d \mathbb{Z}Q_{\chi_i} + P_j)/P_j$ and $F_{\chi_i}^d = A_{1, \chi_i}^d \oplus \cdots \oplus A_{s, \chi_i}^d$, where $Q_{\chi_i} = \{q \in Q\colon \chi_i(q) \geq 0\}$. By Lemma 3.2 since the ring $\mathbb{Z}Q_{\chi_i}$ is Noetherian the decreasing filtration $\{F_{\chi_i}^d\}_{d \in \mathbb{Z}}$ of $\mathbb{Z}Q_{\chi_i}$-submodules of N can be extended to a decreasing filtration of $\mathbb{Z}Q_{\chi_i}$-submodules of A. We write $\{F_{\chi_i}^d\}_{d \in \mathbb{Z}}$ for the new filtration as well. Finally we define a decreasing filtration $\{F^d\}_{d \in \mathbb{Z}}$ of A by $F^d = \bigcap_{i=1}^m F_{\chi_i}^d$. Since $Q'' \subseteq \operatorname{Ker} \chi_i$ for all $1 \leq i \leq m$, each F^d is a $\mathbb{Z}Q''$-submodule of A.

We define

$$Q'_k = \left\{ q \in Q' \colon \chi_i(q) \geq -k \chi_i(q_{\chi_i}), \, 1 \leq i \leq m \right\}, \qquad S_k = Q'_k \setminus Q'_{k-1} \tag{1}$$

for any positive integer k and $Q'_0 = S_0 = \{1_Q\}$. Thus $Q'_k = \bigcup_{j \leq k} S_j$. Denote $(Q'_k)^{-1} = \{q \in Q' \colon q^{-1} \in Q'_k\}$ and $S_k^{-1} = \{q \in Q' \colon q^{-1} \in S_k\}$.

Let $\pi\colon Q \to G$ be a lifting of the projection map $G \to Q$. We define a path γ in $(Q'_k)^{-1}$ from 1_Q to an element q of $(Q'_k)^{-1}$ to be any sequence

$$\left(1_Q, q_{\chi_{i(1)}}^{\epsilon(1)}, q_{\chi_{i(2)}}^{\epsilon(2)} q_{\chi_{i(1)}}^{\epsilon(1)}, \ldots, \prod_{r=j}^{1} q_{\chi_{i(r)}}^{\epsilon(r)} = q \right)$$

of elements of $(Q'_k)^{-1}$, where j is a positive integer, $\epsilon(r) = \pm 1$, $i(r) \in \{1, 2, \ldots, m-1\}$. For any such path γ there is a corresponding sequence

$$\Big(1_G, \pi(q_{\chi_{i(1)}})^{\epsilon(1)}, \pi(q_{\chi_{i(2)}})^{\epsilon(2)} \pi(q_{\chi_{i(1)}})^{\epsilon(1)}, \ldots,$$

$$\pi(q_{\chi_{i(j)}})^{\epsilon(j)} \pi(q_{\chi_{i(j-1)}})^{\epsilon(j-1)} \cdots \pi(q_{\chi_{i(1)}})^{\epsilon(1)} \Big)$$

of elements of the group G. We write $l(\gamma)$ for the last element of the above sequence and define $W_{k,q}$ to be the set of all elements $l(\gamma)$ when γ runs through all paths in $(Q'_k)^{-1}$ from 1_Q to q. Note that $l(\gamma)$ is an element of G such that $l(\gamma)A = q$ and that

$$W_{k,q} = l(\gamma) W_{k,1_Q}. \tag{2}$$

A similar definition of a path could be given with the modified condition that $i(r) \in \{1, 2, \dots, m\} \setminus \{j\}$ for some $j \le m$ or even that $i(r) \in \{1, 2, \dots, m\}$. We preferred using in the definition of a path the condition that $i(r) \in \{1, 2, \dots, m-1\}$ because $\{q_{\chi_1}, \dots, q_{\chi_{m-1}}\}$ is a $\mathbb{Z}$-basis of Q'.

The following lemma is the only place we use the hypothesis that either G is a split extension of A by Q or A is of finite exponent.

LEMMA 3.3. *$W_{k,1_Q}$ is a finite subgroup of A.*

Proof. Note that any element of $W_{k,1_Q}$ corresponds to a closed path at 1_Q in $(Q'_k)^{-1}$ and the group of closed paths at 1_Q in $(Q'_k)^{-1}$, with group operation the concatenation of paths, is finitely generated. Thus $W_{k,1_Q}$ is a finitely generated subgroup of A. If G is a split extension of A by Q we can assume that π is an embedding of Q in G and hence $W_{k,1_Q}$ contains only the trivial element in A.

If G is non-split then, since by assumption A is a torsion abelian group, $W_{k,1_Q}$ is a finite subgroup of A as required.

We shall prove in Section 3.3 that there exist positive integers k_0 and d such that

$$S_k \subseteq \bigcup_{s \in S_{k+1}} O(s, d) \qquad \text{for all } k \ge k_0, \tag{3}$$

where $O(s, d)$ is the set of all elements in Q' contained in the open ball in $\mathbb{R} \otimes Q' \simeq \mathbb{R}^{m-1}$ with radius d and centre s. Furthermore we will show in Section 3.3 that there exist positive integers n_0 and t_0 and some subsets U_{k+t_0} of S_{k+t_0} such that

$$S_k \subseteq \bigcup_{u \in U_{k+t_0}} O(u, n_0) \qquad \text{for all } k \ge k_0. \tag{4}$$

Denote by $\alpha_{k,q}$ the sum of the elements of $W_{k,q} \subset G$ in the group algebra $\mathbb{Z}G$. We write the elements of the group algebra $\mathbb{Z}G$ as $\mathbb{Z}$-linear combinations of $\pi(q_1)\pi(q_2)\cdots\pi(q_j)T^a = T^{(q_1 q_2 \cdots q_j)a}\pi(q_1)\pi(q_2)\dots\pi(q_j)$ for $q_1, q_2, \dots, q_j \in Q$, $a \in A$ (these expressions are not unique). We define

$$\alpha_k = \sum_{q \in (Q'_k)^{-1}} \alpha_{k,q} \prod_{u \in U_{k+t_0}} (T^u - 1) \in \mathbb{Z}G,$$

where $U_{k+t_0} \subseteq Q$ embeds in $N = \mathbb{Z}Q/P_1 \oplus \cdots \oplus \mathbb{Z}Q/P_s \subseteq A$ via the diagonal map. Set

$$\alpha = (\alpha_k)_{k \geq k_1} \in \mathbb{Z}G^{\mathbb{N}}, \tag{5}$$

where k_1 is a sufficiently big positive integer; we will define it at the beginning of Section 3.4.

LEMMA 3.4. *Provided* (3) *and* (4) *hold and k_1 is a sufficiently big positive integer the image β of α in $E = (\mathbb{Z}G^{\mathbb{N}})_A$ is Q'-invariant.*

Proof. By (2)

$$\alpha_{k,q} = l(\gamma) \sum_{a \in W_{k,1_Q}} T^a$$

for any path γ in $(Q'_k)^{-1}$ from 1_Q to q. If $q \in (Q'_k)^{-1} \cap q_{\chi_i}^{-1}(Q'_k)^{-1}$ for some i between 1 and $m-1$ then

$$\pi(q_{\chi_i})\alpha_{k,q} = \pi(q_{\chi_i})l(\gamma) \sum_{a \in W_{k,1_Q}} T^a = l(\gamma') \sum_{a \in W_{k,1_Q}} T^a = \alpha_{k,q_{\chi_i}q},$$

where γ' is the concatenation of the path γ from 1_Q to q and the path (q, qq_{χ_i}) and both paths are in $(Q'_k)^{-1}$.

Then for every $1 \leq i \leq m-1$ one calculates

$$\begin{aligned}
\pi(q_{\chi_i})\alpha_k - \alpha_k &= \sum_{q \in (Q'_k)^{-1}} \pi(q_{\chi_i})\alpha_{k,q} \prod_{u \in U_{k+t_0}} (T^u - 1) \\
&\quad - \sum_{q \in (Q'_k)^{-1}} \alpha_{k,q} \prod_{u \in U_{k+t_0}} (T^u - 1) \\
&= \sum_{q \in (Q'_k)^{-1} \cap q_{\chi_i}^{-1}(Q'_k)^{-1}} \alpha_{k,q_{\chi_i}q} \prod_{u \in U_{k+t_0}} (T^u - 1) \\
&\quad + \sum_{q \in (Q'_k)^{-1} \setminus q_{\chi_i}^{-1}(Q'_k)^{-1}} \pi(q_{\chi_i})\alpha_{k,q} \prod_{u \in U_{k+t_0}} (T^u - 1) \\
&\quad - \sum_{q \in (Q'_k)^{-1}} \alpha_{k,q} \prod_{u \in U_{k+t_0}} (T^u - 1) \\
&= \sum_{q \in (Q'_k)^{-1} \setminus q_{\chi_i}^{-1}(Q'_k)^{-1}} \pi(q_{\chi_i})\alpha_{k,q} \prod_{u \in U_{k+t_0}} (T^u - 1) \\
&\quad - \sum_{q \in (Q'_k)^{-1} \setminus q_{\chi_i}(Q'_k)^{-1}} \alpha_{k,q} \prod_{u \in U_{k+t_0}} (T^u - 1).
\end{aligned} \tag{6}$$

Consider the element $\pi(q_{\chi_i})\alpha_{k,q}\prod_{u\in U_{k+t_0}}(T^u-1)$ for some $q \in (Q'_k)^{-1} \setminus q_{\chi_i}^{-1}(Q'_k)^{-1}$ and $k \geq k_1$. Then $q^{-1} \in Q'_k \setminus q_{\chi_i} Q'_k \subseteq S_k \cup \ldots \cup S_{k-b}$ for some sufficiently big positive number b not depending on k and by (3) there exists an element $s \in S_k$ such that $q^{-1} \in O(s, bd)$ if $k \geq k_1 \geq k_0 + b$. By (4) there exists an element u_0 in U_{k+t_0} such that $s \in O(u_0, n_0)$. Then $qu_0 \in O(1_Q, n_0 + bd)$ and

$$\pi(q_{\chi_i})\alpha_{k,q}\prod_{u\in U_{k+t_0}}(T^u-1) = \prod_{u\in U_{k+t_0}}(T^{q_{\chi_i}qu}-1)\pi(q_{\chi_i})\alpha_{k,q}$$

$$\in \sum_{r\in q_{\chi_i}O(1_Q,\, n_0+bd)}(T^r-1)\mathbb{Z}G \tag{7}$$

We remind the reader that the elements of Q are viewed as elements of $N \subseteq A$ via the diagonal map from $\mathbb{Z}Q$ to $N = \mathbb{Z}Q/P_1 \oplus \cdots \oplus \mathbb{Z}Q/P_s$.

Now we consider $\alpha_{k,q}\prod_{u\in U_{k+t_0}}(T^u-1)$ for $q \in (Q'_k)^{-1} \setminus q_{\chi_i}(Q'_k)^{-1}$ for some $k \geq k_1$. Then $q^{-1} \in Q'_k \setminus q_{\chi_i}^{-1} Q'_k$ and

$$Q'_k \setminus q_{\chi_i}^{-1} Q'_k \subseteq S_k \cup S_{k-1} \cup \cdots \cup S_{k-b} \tag{8}$$

for some positive integer b not depending on k but probably depending on i. Choosing b sufficiently big we can assume that (8) holds for all $1 \leq i \leq m$. Assume further that $k_1 \geq b + k_0$ and hence $k \geq b + k_0$. Then $q^{-1} \in S_{k-j}$ for some $0 \leq j \leq b$ and by (3) there exists an element s in S_k such that $q^{-1} \in O(s, jd) \subseteq O(s, bd)$. Now by (4) we have an element u_0 in U_{k+t_0} such that $s \in O(u_0, n_0)$ and hence $qu_0 \in O(1_Q, bd + n_0)$.
Then one calculates

$$\alpha_{k,q}\prod_{u\in U_{k+t_0}}(T^u-1) = \prod_{u\in U_{k+t_0}}(T^{qu}-1)\alpha_{k,q} \in \sum_{r\in O(1_Q,\, bd+n_0)}(T^r-1)\mathbb{Z}G \tag{9}$$

By (6), (7), and (9) we deduce that the image β of α in $(\mathbb{Z}G^{\mathbb{N}})_A$ is q_{χ_i}-invariant for $1 \leq i \leq m-1$ and hence it is Q'-invariant.

3.3. *The Choice of k_0, t_0, and U_k*

In this section we find sets U_k and positive integers k_0, d, t_0 which satisfy (3) and (4) from Section 3.2.

Let A and B be the subsets of $\mathbb{R}^s$ for some positive integer s. We define $\delta(A, B)$ to be the infimum of all real numbers d such that for any element $a \in A$ there exists an element $b \in B$ with the property that the distance between a and b is not more than d. Note that in general $\delta(A, B) \neq \delta(B, A)$. Later we will need the following simple fact.

LEMMA 3.5. *Let s be a natural number and let $l_1, l_2, \ldots, l_{s+1}$ be linear functionals on $\mathbb{R}^s$ such that their convex hull in* $\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}) \simeq \mathbb{R}^s$ *is an s-simplex with the origin as an internal point. Suppose that for some real numbers $a_i \geq b_i$, $1 \leq i \leq s+1$ the sets $A = \{r \in \mathbb{R}^s \mid l_i(r) \geq a_i$* for all *$1 \leq i \leq s+1\}$ and $B = \{r \in \mathbb{R}^s \mid l_i(r) \geq b_i$* for all *$1 \leq i \leq s+1\}$ are s-simplexes. Denote the sets of vertices of A and B by A^o and B^o, respectively, so $A^o = \{a_1^o, \ldots, a_{s+1}^o\}$, $B^o = \{b_1^o, \ldots, b_{s+1}^o\}$ where $l_i(a_j^o) = a_i$ for $1 \leq i \neq j \leq s+1$ and $l_i(b_j^o) = b_i$ for all $1 \leq i \neq j \leq s+1$. Then $A \subseteq B$ and*

$$\delta(B, A) \leq \max\{\|a_i^o - b_i^o\| \colon 1 \leq i \leq s+1\}. \tag{1}$$

If in addition $0 \in A$ and $B = r_0 A$ for some real number $r_0 > 1$, we have

$$\delta(\overline{B \setminus A}, \delta B) \leq \max\{\|a_i^o - b_i^o\| \colon 1 \leq i \leq s+1\}, \tag{2}$$

where δB denotes the boundary of B and $\overline{B \setminus A}$ is the closure of $B \setminus A$.

Remark. Inequality (2) is true without the additional hypothesis but we need it only in the described form.

Proof. (1) Let b be an element of B and $a \neq b$ an element of $A \setminus \delta B$. Then the intersection of the line through a and b and the boundary δB of B contains only two distinct points, say b_1 and b_2. Without loss of generality we can assume that b is a point between a and b_2 and so $b = ta + (1-t)b_2$ for some $t \in [0,1)$. Since $b_2 \in \delta B$ there exists some $i \in \{1, 2, \ldots, s+1\}$ and non-negative real numbers $\{\mu_j\}_{1 \leq j \neq i \leq s+1}$ such that $b_2 = \sum_{1 \leq j \neq i \leq s+1} \mu_j b_j^o$ and $\sum_{1 \leq j \neq i \leq s+1} \mu_j = 1$. We define $c = ta + (1-t)\sum_{1 \leq j \neq i \leq s+1} \mu_j a_j^o$. Since A is a convex set c is an element of A and $\|b - c\| = \|(1-t)(b_2 - \sum_{j \neq i} \mu_j a_j^o)\| \leq \|\sum_{j \neq i} \mu_j (b_j^o - a_j^o)\| \leq \max\{\|b_j^o - a_j^o\| \colon 1 \leq j \leq s+1\}$.

(2) Let b be an element of $B \setminus A$ and let the ray from 0 through b intersect the boundary δA of A at the point c_1 and the boundary δB of B at the point c_2. Then there exists some $i \in \{1, 2, \ldots, s+1\}$ and non-negative real numbers $\{\mu_j\}_{1 \leq j \neq i \leq s+1}$ such that $\sum_{j \neq i} \mu_j = 1$, $c_1 = \sum_{j \neq i} \mu_j a_j^o$, and $c_2 = \sum_{j \neq i} \mu_j b_j^o$. Then $\|b - c_2\| \leq \|c_1 - c_2\| = \|\sum_{j \neq i} \mu_j (b_j^o - a_j^o)\| \leq \max\{\|b_j^o - a_j^o\| \colon 1 \leq j \leq s+1\}$.

Then $\delta(\overline{B \setminus A}, \delta B) = \delta(B \setminus A, \delta B) \leq \max\{\|a_i^o - b_i^o\| \colon 1 \leq i \leq s+1\}$, as required.

The following lemma is an easy consequence of Lemma 3.5.

LEMMA 3.6. *Let s be a natural number and let $l_1, l_2, \ldots, l_{s+1}$ be linear functionals on $\mathbb{R}^s$ such that their convex hull in* $\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}) \simeq \mathbb{R}^s$ *is an s-simplex containing the origin as an internal point. Let $A_k = \{r \in \mathbb{R}^s \mid l_i(r) \geq a_{i,k},\ 1 \leq i \leq s+1\}$ for some real numbers $a_{i,k}$ and $B_k = \{r \in \mathbb{R}^s \mid l_i(r) \geq a_{i,k} - b_i,\ 1 \leq i \leq s+1\}$ for some positive real numbers b_i, not depending* *on k, be s-simplexes for all $k \in \mathbb{N}$. Then $A_k \subset B_k$ and*

$$\text{the set } \{\delta(B_k, A_k)\}_{k \in \mathbb{N}} \text{ is bounded.} \tag{3}$$

If in addition for all k the origin belongs to A_k and $B_k = r_k A_k$ for some real number $r_k > 1$ then

$$\text{the set } \{\delta(\overline{B_k \setminus A_k}, \delta B_k)\}_{k \in \mathbb{N}} \text{ is bounded.} \tag{4}$$

Proof. Let $A_k^o = \{a_{1,k}^o, a_{2,k}^o, \ldots, a_{s+1,k}^o\}$ and $B_k^o = \{b_{1,k}^o, b_{2,k}^o, \ldots, b_{s+1,k}^o\}$ be the vertices of A_k and B_k, respectively, given by $l_i(a_{j,k}^o) = a_{i,k}$ for $1 \le i \ne j \le s+1$ and $l_i(b_{j,k}^o) = a_{i,k} - b_i$ for $1 \le i \ne j \le s+1$. Then $l_i(a_{j,k}^o - b_{j,k}^o) = b_i$ for $1 \le i \ne j \le s+1$ does not depend on k, so $a_{j,k}^o - b_{j,k}^o$ does not depend on k. By (1) $\delta(B_k, A_k) \le \max\{\|a_{j,k}^o - b_{j,k}^o\|: 1 \le j \le s+1\}$, so (3) follows immediately. Similarly (4) follows from (2).

By the choice of the characters $\chi_1, \ldots, \chi_m$ for any $1 \le i \le m$ there exists an integer $j(i)$ such that

$$\text{the ring } B_i = (\mathbb{Z}Q_{\chi_i} + P_{j(i)})/(q_{\chi_i}\mathbb{Z}Q_{\chi_i} + P_{j(i)}) \text{ is non-trivial.} \tag{5}$$

Let J_i be a maximal ideal in B_i. Then B_i/J_i is a finite field and there exists a positive integer r_i such that

$$(\mathrm{Ker}(\chi_i))^{r_i} = 1 \qquad \text{in } B_i/J_i. \tag{6}$$

We define

$$L_{i,k} = \{q \in S_k : \chi_i(q) = -k\chi_i(q_{\chi_i})\} \subset Q',$$

$$W_{i,k} = \{q \in L_{i,k} \mid \chi_j(q) \ge -(k-1)\chi_j(q_{\chi_j}) \text{ for } 1 \le j \ne i \le m\},$$

$$U_{i,k} = \{q \in W_{i,k} \mid qq_{\chi_i}^k \in (\mathrm{Ker}\,\chi_i)^{r_i}\}, \text{ and set } U_k = \textstyle\bigcup_{i=1}^m U_{i,k}. \tag{7}$$

We define some subsets of $\mathbb{R}^{m-1} = Q' \otimes \mathbb{R}$ by

$$Q_k'^* = \{r \in \mathbb{R}^{m-1} \mid (q_{\chi_j}, r) \ge -k\chi_j(q_{\chi_j}) \text{ for all } 1 \le j \le m\},$$

$$S_k^* = Q_k'^* \setminus Q_{k-1}'^*,$$

$$L_{i,k}^* = \{r \in \mathbb{R}^{m-1} \mid (q_{\chi_j}, r) \ge -k\chi_j(q_{\chi_j}) \text{ for all } 1 \le j \ne i \le m,$$
$$(q_{\chi_i}, r) = -k\chi_i(q_{\chi_i})\},$$

$$W_{i,k}^* = \{r \in \mathbb{R}^{m-1} \mid (q_{\chi_j}, r) \ge -(k-1)\chi_j(q_{\chi_j}),\ 1 \le j \ne i \le m,$$
$$(q_{\chi_i}, r) = -k\chi_i(q_{\chi_i})\}.$$

Then $W_{i,k} \subset L_{i,k} \subset S_k \subset Q'_k$ are the points with integral coordinates in $W^*_{i,k} \subset L^*_{i,k} \subset S^*_k \subset Q'^*_k \subset \mathbb{R}^{m-1}$, where Q' is identified with $\mathbb{Z}^{m-1}$ in $\mathbb{R}^{m-1}$.

The following lemma ensures the existence of positive integers k_0 and d satisfying (3) in Section 3.2.

LEMMA 3.7. *There exist positive integers k_0 and d such that*

$$S_k \subseteq \bigcup_{s \in S_{k+1}} O(s,d) \qquad \text{for all } k \geq k_0.$$

Proof. By (4) the sets $\{\delta(Q^*_k \setminus Q^*_{k-1}, \delta(Q^*_k))\}_{k \in \mathbb{N}}$ and $\{\delta(\overline{Q^*_{k+1} \setminus Q^*_k}, \delta(Q^*_{k+1}))\}_{k \in \mathbb{N}}$ are bounded. Since the boundary $\delta(Q^*_k)$ of Q^*_k lies in $\overline{Q^*_{k+1} \setminus Q^*_k}$ we can deduce that the set $\{\delta(S^*_k, \delta(Q^*_{k+1}))\}_{k \in \mathbb{N}}$ is bounded as well.

We claim that for sufficiently large k, say $k \geq k_0$, the set $L_{i,k}$ is non-empty and the set $\{\delta(L^*_{i,k}, L_{i,k})\}_{k \geq k_0}$ is bounded. If this is the case then using that $\delta(Q^*_{k+1}) = \bigcup_{i=1}^m L^*_{i,k+1}$ and $\bigcup_{i=1}^m L_{i,k+1} \subseteq S_{k+1}$ we get the set $\{\delta(S^*_k, S_{k+1})\}_{k \geq k_0 - 1}$ is bounded and S_k is nonempty for $k \geq k_0$. Consequently there exists a positive integer d such that $\delta(S_k, S_{k+1}) < d$ for all $k \geq k_0$, as required in Lemma 3.7.

Define $B_{i,k} = q^k_{\chi_i} L^*_{i,k} \subseteq (\operatorname{Ker} \chi_i \cap Q') \otimes \mathbb{R} \subset Q' \otimes \mathbb{R} = \mathbb{R}^{m-1}$. Since χ_i is a discrete character there exists a positive integer z_i such that any closed $(m-2)$-ball in $(\operatorname{Ker} \chi_i \cap Q') \otimes \mathbb{R}$ with radius bigger or equal to z_i contains a point from the free abelian group $Q' \cap \operatorname{Ker} \chi_i$. Define $A_{i,k}$ to be the set of elements r in $B_{i,k}$ such that the $(m-2)$-ball in $(\operatorname{Ker} \chi_i \cap Q') \otimes \mathbb{R}$ with centre r and radius z_i lies in $B_{i,k}$. Since the maximal radius of an $(m-2)$-ball entirely contained in $L^*_{i,k}$ goes to infinity as k goes to infinity, for sufficiently big k, say $k \geq k_0$, $A_{i,k}$ is an $(m-2)$-simplex. Note that by (3) the set $\{\delta(B_{i,k}, A_{i,k})\}_{k \geq k_0}$ is bounded. Then for $k \geq k_0$ we have $\delta(L^*_{i,k}, L_{i,k}) = \delta(q^k_{\chi_i} L^*_{i,k}, q^k_{\chi_i} L_{i,k}) \leq \delta(B_{i,k}, A_{i,k}) + \delta(A_{i,k}, q^k_{\chi_i} L_{i,k}) \leq \delta(B_{i,k}, A_{i,k}) + z_i$ and hence $\{\delta(L^*_{i,k}, L_{i,k})\}_{k \geq k_0}$ is bounded, as required.

The following lemma about the filtration $\{F^d\}_{d \in \mathbb{Z}}$ defined at the beginning of Section 3.2 will play a key role in the proof that the image of the element β in $(\mathbb{Z}G^{\mathbb{N}})_{G''}$ is nontrivial.

LEMMA 3.8. *For sufficiently large positive integer t_0 we have*

$$W_{k,1_Q} \subseteq F^{-k-t_0+1} \tag{8}$$

for all k.

Proof. Let Γ be the set of all closed paths in Q' of the type

$$\left(1_Q, q_{\chi_i}^{\epsilon(i)}, q_{\chi_i}^{\epsilon(i)} q_{\chi_j}^{\epsilon(j)}, q_{\chi_j}^{\epsilon(j)}, 1_Q\right),$$

where $1 \leq i \neq j \leq m-1$ and $\epsilon(i), \epsilon(j) \in \{-1, 1\}$. Then $W_{k, 1_Q}$ is an additive subgroup of A contained in the subgroup generated by $l(\gamma_1)^{-1} l(\gamma) l(\gamma_1)$, where γ_1 is a path in $(Q'_k)^{-1}$ starting from 1_Q and $\gamma \in \Gamma$. Since A is a left $\mathbb{Z}Q$-module via conjugation and all the paths γ_1 lie in $(Q'_k)^{-1}$ we have

$$W_{k, 1_Q} \subseteq \sum_{\gamma \in \Gamma} \mathbb{Z} Q'_k l(\gamma).$$

Via the diagonal map from Q to N the set Q'_k maps to elements in F^{-k} and hence it is sufficient to choose t_0 such that $l(\gamma) \in F^{-t_0+1}$ for all $\gamma \in \Gamma$; this is possible because Γ is a finite set.

The following proposition ensures the existence of natural numbers n_0 and t_0 satisfying (4) from Section 3.2. The statement is geometrically clear but the precise calculation involves several simple geometrical arguments.

PROPOSITION 3.9. *There exist a positive integer t_0 satisfying* (8) *and a natural number n_0, depending on t_0, such that*

$$S_k \subseteq \bigcup_{u \in U_{k+t_0}} O(u, n_0) \qquad \text{for all } k \geq k_0.$$

Proof. (1) We claim that there exists a positive integer d_0 such that the set U_k is non-empty for all $k \geq d_0$ and the set $\{\delta(S_k^*, U_k)\}_{k \geq d_0}$ is bounded.

We have seen in the proof of Lemma 3.7 that the set $\{\delta(S_k^*, \bigcup_{i=1}^m L_{i,k}^*)\}_{k \in \mathbb{N}}$ is bounded. Now $W_{i,k}^* \subset L_{i,k}^*$ are $(m-2)$-simplexes and we can apply (3) to the $(m-2)$-simplexes $q_{\chi_i}^k W_{i,k}^* \subset q_{\chi_i}^k L_{i,k}^* \subset (\operatorname{Ker} \chi_i \cap Q') \otimes \mathbb{R}$, for $k \in \mathbb{N}$. Since $\delta(L_{i,k}^*, W_{i,k}^*) = \delta(q_{\chi_i}^k L_{i,k}^*, q_{\chi_i}^k W_{i,k}^*)$ we have that the set $\{\delta(L_{i,k}^*, W_{i,k}^*)\}_{k \in \mathbb{N}}$ is bounded.

It remains to prove that for k sufficiently large, say $k \geq d_0$, $U_{i,k}$ is nonempty-for all $1 \leq i \leq m$ and the set $\{\delta(W_{i,k}^*, U_{i,k})\}_{1 \leq i \leq m;\ k \geq d_0}$ is bounded. If this is the case we can deduce that there exists a positive integer n_1 such that

$$U_k \text{ is non-empty and } \delta(S_k^*, U_k) < n_1 \text{ for all } k \geq d_0. \tag{9}$$

We consider $B_{i,k} = q_{\chi_i}^k W_{i,k}^* = \{q_{\chi_i}^k q \mid q \in W_{i,k}^*\}$, an $(m-2)$-simplex in $(\operatorname{Ker} \chi_i \cap Q') \otimes \mathbb{R}$. Then $q_{\chi_i}^k U_{i,k} = B_{i,k} \cap (\operatorname{Ker} \chi_i \cap Q')^{r_i} \subseteq Q' \simeq \mathbb{Z}^{m-1}$. Since χ_i is a discrete character the subgroup $Q_i = Q' \cap \operatorname{Ker} \chi_i$ of Q' is a free abelian of rank $m-2$ and hence there exists a positive integer z_i such that any closed $(m-2)$-ball in $Q_i \otimes \mathbb{R} \subset Q' \otimes \mathbb{R}$, with radius bigger than or equal to z_i, contains a point of $(Q_i)^{r_i}$.

Let $A_{i,k}$ be the set of all elements $r \in B_{i,k}$ such that the closed $(m-2)$-ball in $Q_i \otimes \mathbb{R}$ with center r and radius z_i lies in $B_{i,k}$; so $A_{i,k}$ is empty, a single point, or an $(m-2)$-simplex contained in $B_{i,k}$. Since $W^*_{i,k}$ is an $(m-2)$-simplex with the property that the maximal radius of a closed $(m-2)$-ball contained in $W^*_{i,k}$ goes to infinity as k goes to infinity, we have that for sufficiently large k, say $k \geq d_0$, $A_{i,k}$ is an $(m-2)$-simplex. Then the set $q^k_{\chi_i} U_{i,k}$ is non-empty for all $k \geq d_0$. By (3) we obtain that the set $\{\delta(B_{i,k}, A_{i,k})\}_{k \geq d_0}$ is bounded. Then $\delta(B_{i,k}, q^k_{\chi_i} U_{i,k}) \leq \delta(B_{i,k}, A_{i,k}) + \delta(A_{i,k}, q^k_{\chi_i} U_{i,k}) \leq \delta(B_{i,k}, A_{i,k}) + z_i$ for $k \geq d_0$. Thus

$$\{\delta(W^*_{i,k}, U_{i,k})\}_{k \geq d_0} = \left\{\delta\left(q^k_{\chi_i} W^*_{i,k}, q^k_{\chi_i} U_{i,k}\right)\right\}_{k \geq d_0} \text{ is bounded}$$

as required.

(2) Let t_0 be any positive number satisfying (8) and $t_0 \geq d_0$. Then by Lemma 3.7 we have

$$S_k \subseteq \bigcup_{u \in S_{k+t_0}} O(u, dt_0) \qquad \text{for all } k \geq k_0. \tag{10}$$

Finally set $n_0 = n_1 + dt_0$, where n_1 is the positive integer given in (9). Using (9) and (10) we obtain

$$S_k \subseteq \bigcup_{u \in S_{k+t_0}} O(u, dt_0) \subseteq \bigcup_{u \in U_{k+t_0}} O(u, n_0) \qquad \text{for all } k \geq k_0$$

as required.

3.4. *The Image of β is Non-trivial in $(\mathbb{Z}G^{\mathbb{N}})_{G''}$*

Now we summarize. We take k_0 and d to be the integers defined in Lemma 3.7, then define the positive integers n_0 and t_0 and the sets U_{k+t_0} to be the ones given by Proposition 3.9, and finally set $k_1 = b + k_0$, where b is the positive integer introduced in (8), Section 3.2. Let α be the element of $\mathbb{Z}G^{\mathbb{N}}$ defined in (5), Section 3.2 with the chosen positive integers k_1 and t_0 and the sets U_{k+t_0}, $k \in \mathbb{N}$. Then both conditions (3) and (4) from Sections 3.2 hold, so by Lemma 3.4 the image β of α in $(\mathbb{Z}G^{\mathbb{N}})_A$ is Q'-invariant.

We remind the reader that G'' was defined in Section 3.1 (see Proposition 3.1) and finally we prove

THEOREM 3.10. *The image of β in $(\mathbb{Z}G^{\mathbb{N}})_{G''}$ is non-trivial.*

Proof. Assume, to the contrary, that the image of β in $(\mathbb{Z}G^{\mathbb{N}})_{G''}$ is trivial. By Proposition 3.1 there exists an integer d^* such that

$$\alpha_{k,1_Q} \prod_{u \in U_{k+t_0}} (T^u - 1) = 0 \qquad \text{in } \mathbb{Z}\left[A/F^{d^*}\right].$$

Note that $\alpha_{k,1_Q}$ was defined before Lemma 3.4 and should not be confused with the similar notation used in Proposition 3.1. By (8) in Section 3.3 we have $\alpha_{k,1_Q} = cT^0$ in $\mathbb{Z}[A/F^{-k-t_0+1}]$ where c is the number of the elements in the finite set $W_{k,1_Q}$; we remind the reader that the set $W_{k,1_Q}$ was defined in Section 3.2. For sufficiently large k, say $k \geq k^*$, $F^{d^*} \subseteq F^{-k-t_0+1}$ and so $\alpha_{k,1_Q}\prod_{u \in U_{k+t_0}}(T^u - 1) = 0$ in $\mathbb{Z}[A/F^{-k-t_0+1}]$. Hence

$$\prod_{u \in U_{k+t_0}} (T^u - 1) = 0 \qquad \text{in } \mathbb{Z}\left[A/F^{-k-t_0+1}\right] \text{for } k \geq k^*. \tag{1}$$

Let V_{k+t_0} be the abelian subgroup of A/F^{-k-t_0+1} generated by the classes of the elements in U_{k+t_0}; we remind the reader that the elements of $U_{k+t_0} \subset Q$ embed in $N \subseteq A$ via the diagonal map. Let $V_{i,k+t_0}$ be the subgroup of V_{k+t_0} generated by the classes of the elements in $U_{i,k+t_0}$. We claim that V_{k+t_0} is a direct sum of its subgroups $V_{i,k+t_0}$ for $1 \leq i \leq m$.

Let λ_i be an element of the abelian subgroup of N generated by $U_{i,k+t_0}$, $1 \leq i \leq m$. By (7) in Section 3.3 we have $\chi_j(U_{i,k+t_0}) \geq -(k + t_0 - 1)\chi_j(q_{\chi_j})$ for $j \neq i$, so $\lambda_i \in F_{\chi_j}^{-k-t_0+1}$ for $1 \leq j \neq i \leq m$. If $\sum_{i=1}^m \lambda_i \in F^{-k-t_0+1} = \bigcap_{i=1}^m F_{\chi_i}^{-k-t_0+1} \subseteq F_{\chi_i}^{-k-t_0+1}$, since $\lambda_j \in F_{\chi_i}^{-k-t_0+1}$ for all $j \neq i$, we have $\lambda_i \in F_{\chi_i}^{-k-t_0+1}$. Using again $\lambda_i \in F_{\chi_j}^{-k-t_0+1}$ for $1 \leq j \neq i \leq m$ we get $\lambda_i \in \bigcap_{j=1}^m F_{\chi_j}^{-k-t_0+1} = F^{-k-t_0+1}$; hence V_{k+t_0} is a direct sum of its subgroups $V_{i,k+t_0}$ for $1 \leq i \leq m$.

One can write (1) of the form $\prod_{u \in U_{k+t_0}}(T^u - 1) = 0$ in $\mathbb{Z}[V_{k+t_0}]$; hence there exists i_0 such that

$$\prod_{u \in U_{i_0,k+t_0}} (T^u - 1) = 0 \qquad \text{in } \mathbb{Z}\left[V_{i_0,k+t_0}\right] \text{ for } k = k^*.$$

Since

$$U_{i_0,k+t_0} \subseteq F_{\chi_j}^{-k-t_0+1} \qquad \text{for } j \neq i_0$$

the abelian group $V_{i_0,k+t_0}$ is isomorphic to the abelian subgroup $V'_{i_0,k+t_0}$ of $A/F_{\chi_{i_0}}^{-k-t_0+1}$ generated by the image of $U_{i_0,k+t_0}$ in N under the diagonal map. Thus

$$\prod_{u \in U_{i_0,k+t_0}} (T^u - 1) = 0 \qquad \text{in } \mathbb{Z}\left[V'_{i_0,k+t_0}\right] \text{ for } k = k^*$$

and

$$V'_{i_0,k+t_0} \text{ embeds in } \bigoplus_{j=1}^{s} \left(q_{\chi_{i_0}}^{-k-t_0}\mathbb{Z}Q_{\chi_{i_0}} + P_j\right)/\left(q_{\chi_{i_0}}^{-k-t_0+1}\mathbb{Z}Q_{\chi_{i_0}} + P_j\right). \quad (2)$$

We know by (5) in Section 3.3 that the ring $B_{i_0} = (\mathbb{Z}Q_{\chi_{i_0}} + P_{j(i_0)})/(q_{\chi_{i_0}}\mathbb{Z}Q_{\chi_{i_0}} + P_{j(i_0)})$ is non-trivial. Obviously $B'_{i_0} := (q_{\chi_{i_0}}^{-k-t_0}\mathbb{Z}Q_{\chi_{i_0}} + P_{j(i_0)})/(q_{\chi_{i_0}}^{-k-t_0+1}\mathbb{Z}Q_{\chi_{i_0}} + P_{j(i_0)})$ is isomorphic to B_{i_0} as a $\mathbb{Z}(\operatorname{Ker}\chi_{i_0})$-module. By (2) we have

$$\prod_{u\in U_{i_0,k+t_0}} (T^u - 1) = 0 \qquad \text{in } \mathbb{Z}\left[B'_{i_0}\right] \text{for } k = k^*$$

and so

$$\prod_{u\in q_{\chi_{i_0}}^{k+t_0}U_{i_0,k+t_0}} (T^u - 1) = 0 \text{ in } \mathbb{Z}\left[B_{i_0}\right] \text{ and hence in } \mathbb{Z}\left[B_{i_0}/J_{i_0}\right] \text{ for } k = k^*,$$

where J_{i_0} is the ideal of B_{i_0} defined in Section 3.3. By (6) in Section 3.3 all the elements of $(\operatorname{Ker}\chi_{i_0})^{r_{i_0}}$ represent $1 + J_{i_0}$ in B_{i_0}/J_{i_0} and by (7) in Section 3.3 $q_{\chi_{i_0}}^{k+t_0}U_{i_0,k+t_0} \subseteq (\operatorname{Ker}\chi_{i_0})^{r_{i_0}}$, so

$$(T^{1+J_{i_0}} - 1)_{|U_{i_0,k+t_0}|} = 0 \qquad \text{in } \mathbb{Z}\left[B_{i_0}/J_{i_0}\right] \text{ for } k = k^*.$$

The last statement contradicts the fact that $1 + J_{i_0}$ is a non-trivial element of the finite field B_{i_0}/J_{i_0} and $\mathbb{Z}[B_{i_0}/J_{i_0}]$ does not have nilpotent elements. This completes the proof of Theorem 3.10.

We have just shown that the image of our special element β under the canonical map

$$H^0\left(Q', (\mathbb{Z}G^{\mathbb{N}})_A\right) \to H_0\left(Q'', (\mathbb{Z}G^{\mathbb{N}})_A\right)$$

is non-trivial. This implies that G is not of type FP_m because as shown in [1] the above map factors through $H_{m-1}(Q, (\mathbb{Z}G^{\mathbb{N}})_A) \simeq H_{m-1}(G, \mathbb{Z}G^{\mathbb{N}})$ and the latter should be trivial if G is of type FP_m. This finishes the proof of Theorem B.

It is worth noting that the proof of Theorem B can be used to show the result of Bieri and Strebel [5] that FP_2 implies 2-tameness. As before we can assume that A is not 2-tame and then the subgroup Q' of Q defined in Section 3.1 has rank one. The latter is crucial to claim existence of non-trivial Q'-invariant elements in $(\mathbb{Z}G^{\mathbb{N}})_A$ (in the notation of Section 3.2, since the rank of Q' is one, for any non-negative integer k the subgroup $W_{k,1_Q}$ of A is trivial), which is the main obstacle in the non-split case.

Finally we remark that in the non-split case the condition that A is of finite exponent was used only in the proof of Lemma 3.3 but the observation that $W_{k,1_Q}$ is finite is vital for the construction of the element α.

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REFERENCES

1. H. Åberg, Bieri-Strebel valuations (of finite rank), *Proc. London Math. Soc.* (3) **52** (1986), 269–304.
2. R. Bieri, Homological dimension of discrete groups, *Queen Mary College Math. Notes* (1981).
3. R. Bieri and J. Groves, Metabelian groups of type $(FP)_\infty$ are virtually of type FP, *Proc. London Math. Soc.* (3) **45** (1982), 365–384.
4. R. Bieri and J. Groves, The geometry of the set of characters induced by valuations, *J. Reine Angew. Math.* **347** (1984), 168–195.
5. R. Bieri and R. Strebel, Valuations and finitely presented metabelian groups, *Proc. London Math. Soc.* (3) **41** (1980), 439–464.
6. R. Bieri and R. Strebel, A geometric invariant for modules over an abelian group, *J. Reine Angew. Math.* **322** (1981), 170–189.
7. R. Bieri and R. Strebel, ‘‘Geometric Invariants for Discrete Groups,’’ in preparation.
8. N. Bourbaki, ‘‘Elements of Mathematics (Commutative Algebra),’’ Hermann, Paris, 1972.
9. K. S. Brown, Finiteness properties of groups, *J. Pure Appl. Algebra* **44** (1987), 45–75.
10. K. U. Bux, Finiteness properties of certain metabelian arithmetic groups in the function field case, submitted for publication.
11. G. A. Noskov, Bieri–Strebel invariant and homological finiteness properties of metabelian groups, SFB-Preprint 93-028, Universität Bielefeld, 1993.